NITHISH'S

MATHEMATICS

10^{th} Std

Strictly based on Text book, prescribed by the Department of Education. Government of TamilNadu.

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Salient Features

- *Key points - formulae and Definitions*
- *Type wise Example and Exercise sums - easy to difficult approach*
- *Detailed steps with Notes for required steps*
- *One Marks with detailed Steps*

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CHAPTER 1

RELATIONS AND FUNCTIONS

Exercise 1.1

KEY POINTS

1. Ordered Pair

A pair of numbers written in a particular order is known as ordered pair of numbers.

Example: (1,1), (2,4), (3,9), (4,16), (5,25).

2. Cartesian Product

If A and B are two non-empty sets, then the set of all ordered pairs (a, b) such that $a \in A, b \in B$ is called the **cartesian product** (or) cross product of A and B, and is denoted by $A \times B$.

$\therefore A \times B = \{ (a, b) \mid a \in A, b \in B \}$

Note

(i) If $a = b$ then $(a, b) = (b, a)$

(ii) In general $A \times B \neq B \times A$, but $n(A \times B) = n(B \times A)$

(iii) $A \times B = \phi$ if only if $A = \phi$ or $B = \phi$

(iv) If $n(A) = P$ and $n(B) = q$ then $n(A \times B) = pq$

(v) Distributive property of cartesian product over union and intersection is given by

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Type: I [Problems based on cartesian product Given sets *A*, *B* and *C*]

Q.No: 1. (i) (ii) (iii) 2, 4, 5, Example 1.1, Example 1.3, 6. (i) (ii) (iii) and 7. (i) (ii)

1. Find $A \times B, A \times A$ and $B \times A$

(i) $A = \{ 2, -2, 3 \}$ and $B = \{ 1, -4 \}$

- $A \times B = \{ 2, -2, 3 \} \times \{ 1, -4 \}$
  $= \{ (2, 1), (2, -4), (-2, 1), (-2, -4), (3, 1), (3, -4) \}$
- $A \times A = \{ 2, -2, 3 \} \times \{ 2, -2, 3 \}$
  $= \{ (2, 2), (2, -2), (2, 3), (-2, 2), (-2, -2), (-2, 3), (3, 2), (3, -2), (3, 3) \}$
- $B \times A = \{ 1, -4 \} \times \{ 2, -2, 3 \}$
  $= \{ (1, 2), (1, -2), (1, 3), (-4, 2), (-4, -2), (-4, 3) \}$

(ii) $A = B = \{ p, q \}$

- $A \times B = \{ p, q \} \times \{ p, q \}$
  $= \{ (p, p), (p, q), (q, p), q, q) \}$
- $A \times A = \{ p, q \} \times \{ p, q \}$
  $= \{ (p, p), (p, q), (q, p), (q, q) \}$
- $B \times A = \{ p, q \} \times \{ p, q \}$
  $= \{ (p, p), (p, q), (q, p), (q, q) \}$

(iii) $A = \{ m, n \}; B = Q$

- $A \times B = \{ \ \}$
- $A \times A = \{ m, n \} \times \{ m, n \}$
  $= \{ (m, m), (m, n), (n, m), (n, n) \}$
- $B \times A = \{ \ \}$

2. Let $A = \{1, 2, 3\}$ and $B = \{ x \mid x$ is a prime number less than $10\}$. Find $A \times B$ and $B \times A$.

$A = \{ 1, 2, 3 \}$

$B = \{ 2, 3, 5, 7 \}$

- $A \times B = \{ 1, 2, 3 \} \times \{ 2, 3, 5, 7 \}$

  $= \{ (1, 2), (1, 3), (1, 5), (1, 7), (2, 2), (2, 3), (2, 5), (2, 7), (3, 2), (3, 3), (3, 5), (3, 7) \}$

- $B \times A = \{ 2, 3, 5, 7 \} \times \{ 1, 2, 3 \}$

  $= \{ (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (5, 1), (5, 2), (5, 3), (7, 1), (7, 2), (7, 3) \}$

---

4. If $A = \{ 5, 6 \}, B = \{ 4, 5, 6 \}, C = \{ 5, 6, 7 \}$ Show that $A \times A = (B \times B) \cap (C \times C)$

- $A \times A = \{ 5, 6 \} \times \{ 5, 6 \}$

  $= \{ (5, 5), (5, 6), (6, 5), (6, 6) \}$ ...(1)

- $B \times B = \{ 4, 5, 6 \} \times \{ 4, 5, 6 \}$

  $= (4, 4), (4, 5), (4, 6),(5, 4), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)$

- $C \times C = \{ 5, 6, 7 \} \times \{ 5, 6, 7 \}$

  $= \{ (5, 5), (5, 6), (5, 7), (6, 5), (6,6), (6, 7), (7, 5), (7, 6), (7, 7) \}$ ...(2)

$(B \times B) \cap (C \times C) = \{ (5, 5), (5, 6), (6, 5), (6, 6) \}$ ...(2)

From (1) & (2) verified.

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5. Given $A = \{ 1, 2, 3 \}, B = \{ 2, 3, 5 \}, C = \{ 3, 4 \}$ and $D = \{ 1, 3, 5 \}$, Check if $(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D)$ is true?

LHS

$A \cap C = \{ 3 \}$

$B \cap D = \{ 3, 5 \}$

$(A \cap C) \times (B \cap D) = \{ 3 \} \times \{ 3, 5 \}$

$= \{ (3, 3), (3, 5) \}$ ...(1)

RHS

$A \times B = \{ 1, 2, 3 \} \times \{ 2, 3, 5 \}$

$= \{ (1, 2), (1, 3), (1, 5), (2, 2), (2, 3), (2, 5), (3, 2), (3, 3), (3, 5) \}$

$C \times D = \{ 3, 4 \} \times \{ 1, 3, 5 \}$

$= \{ (3, 1), (3, 3), (3, 5), (4, 1), (4, 3), (4, 5) \}$

$(A \times B) \cap (C \times D) = \{ (3, 3), (3, 5) \}$ ...(2)

From (1) & (2)

$(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D)$ is true.

Example 1.1

If $A = \{ 1, 3, 5 \}$ and $B = \{ 2, 3 \}$ then find

(i) $A \times B$ and $B \times A$

(ii) Is $A \times B = B \times A$? If not why?

(iii) Show that

$$n (A \times B) = n (B \times A) = n (A) \times n (B)$$

(i) $A \times B = \{ 1, 3, 5 \} \times \{ 2, 3 \}$

$= \{ (1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3) \}$

$B \times A = \{ 2, 3 \} \times \{ 1, 3, 5 \}$

$= \{ (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5) \}$

(ii) $A \times B \neq B \times A$ as $(2, 2) \neq (2, 1)$ and $(1, 3) \neq (3, 1)$ etc.

(iii) We have $n (A) = 3, n (B) = 2,$

$n (A \times B) = 6, n (B \times A) = 6$

$n (A) \times n (B) = 3 \times 2$

$= 6$

Hence

$n (A \times B) = n (B \times A) = n (A) \times n (B)$

---

6. Let $A = \{ x \in W \mid x < 2 \}$. $B = \{ x \in N \mid 1 < x \leq 4 \}$ and $C = \{ 3, 5 \}$

Verify that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Here $A = \{ 0, 1 \}$

$B = \{ 2, 3, 4 \}$

$C = \{ 3, 5 \}$

LHS

$B \cup C = \{ 2, 3, 4, 5 \}$

$A \times (B \cup C) = \{ 0, 1 \} \times \{ 2, 3, 4, 5 \}$

$= \{ (0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5) \}$ ...(1)

RHS

$A \times B = \{ 0, 1 \} \times \{ 2, 3, 4 \}$

$= \{ (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4) \}$

$A \times C = \{ 0, 1 \} \times \{ 3, 5 \}$

$= \{ (0, 3), (0, 5), (1, 3), (1, 5) \}$

$\therefore (A \times B) \cup (A \times C) = \{ (0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5) \}$ ...(2)

From (1) & (2)

$A \times (B \cup C) = (A \times B) \cup (A \times C)$ is verified.

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

LHS

$B \cap C = \{ 3 \}$

$A \times (B \cap C) = \{ 0, 1 \} \times \{ 3 \}$

$= \{ (0, 3), (1, 3) \}$ ...(1)

RHS

$A \times B = \{ 0, 1 \} \times \{ 2, 3, 4 \}$

$= \{ (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4) \}$

$A \times C = \{ 0, 1 \} \times \{ 3, 5 \}$

$= \{ (0, 3), (0, 5), (1, 3), (1, 5) \}$

$(A \times B) \cap (A \times C) = \{ (0, 3), (1, 3) \}$ ...(2)

From (1) & (2)

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ is verified

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(iii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

LHS

$A \cup B = \{ 0, 1, 2, 3, 4 \}$

$(A \cup B) \times C = \{ 0, 1, 2, 3, 4 \} \times \{ 3, 5 \}$

$= \{ (0, 3), (0, 5), (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5), (4, 3), (4, 5) \}$ ...(1)

RHS

$A \times C = \{ 0, 1 \} \times \{ 3, 5 \}$

$= \{ (0, 3), (0, 5), (1, 3), (1, 5) \}$

$B \times C = \{ 2, 3, 4 \} \times \{ 3, 5 \}$

$= \{ (2, 3), (2, 5), (3, 3), (3, 5), (4, 3), (4, 5) \}$

$(A \times C) \cup (B \times C) = \{ (0, 3), (0, 5), (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5), (4, 3), (4, 5) \}$ ...(2)

From (1) & (2)

$(A \cup B) \times C = (A \times C) \cup (B \times C)$ is verified.

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7. Let $A =$ The set of all natural numbers less than 8, $B =$ The set of all prime numbers less than 8, $C =$ The set of even prime number. Verify that

(i) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(ii) $A \times (B - C) = (A \times B) - (A \times C)$

Here

$A = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$B = \{ 2, 3, 5, 7 \}$

$C = \{ 2 \}$

(i) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

LHS

$A \cap B) = \{ 2, 3, 5, 7 \}$

$(A \cap B) \times C = \{ 1, 2, 3, 5, 7 \} \times \{ 2 \}$

$= \{ (2, 2), (3, 2), (5, 2), (7, 2) \}$ ...(1)

RHS

$A \times C = \{ 1, 2, 3, 4, 5, 6, 7 \} \times \{ 2 \}$

$= \{ (1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (7, 2) \}$

$B \times C = \{ 2, 3, 5, 7 \} \times \{ 2 \}$

$= \{ (2, 3), (3, 2), (5, 2), (7, 2) \}$

$(A \times C) \cap (B \times C) = \{ (2, 2), (3, 2), (5, 2), (7, 2) \}$ ...(2)

From (1) & (2)

$(A \cup B) \times C = (A \times C) \cup (B \times C)$ is verified.

(ii) $A \times (B - C) = (A \times B) - (A \times C)$

LHS

$B - C = \{ 2, 3, 5, 7 \} - \{ 2 \}$

$= \{ 3, 5, 7 \}$

$A \times (B - C) = \{ 1, 2, 3, 4, 5, 6, 7 \} \times \{ 3, 5, 7 \}$

$= \{ (1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7)$

$(3, 3), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7),$

$(5, 3), (5, 5), (5, 7), (6, 3), (6, 5), (6, 7)$

$(7, 3), (7, 5), (7, 7) \}$ ...(1)

RHS

$A \times B = \{ 1, 2, 3, 4, 5, 6, 7 \} \times \{ 2, 3, 5, 7 \}$

$= \{ (1, 2), (1, 3), (1, 5), (1, 7), (2, 2), (2, 3),$

$(2, 5), (2, 7), (3, 2), (3, 3), (3, 5), (3, 7),$

$(4, 2), (4, 3), (4, 5), (4, 7), (5, 2), (5, 3),$

$(5, 5), (5, 7)\ (6, 3), (6, 5), (6, 7), (7, 3),$

$(7, 5), (7, 7) \}$

$A \times C = \{ 1, 2, 3, 4, 5, 6, 7 \} \times \{ 2 \}$

$= \{ (1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (7, 2) \}$

$(A \times B) - (A \times C) = \{ (1, 3), (1, 5), (1, 7), (2, 3),$

$(2, 5), (2, 7), (3, 3), (3, 5), (3, 7), (4, 3), (4, 5),$

$(4, 7), (5, 3), (5, 5), (5, 7), (6, 3), (6, 5), (6, 7),$

$(7, 3), (7, 5), (7, 7) \}$ ...(2)

From (1) & (2)

$A \times (B - C) = (A \times B) - (A \times C)$ is verified.

Example 1.3

Let $A = \{ x \in N \mid 1 < x < 4 \}$,

$B = \{ x \in W \mid 0 \leq x < 2 \}$ and

$C = \{ x \in N \mid x < 3 \}$ then verify that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C$

Here $A = \{ 2, 3 \}$

$B = \{ 0, 1 \}$

$C = \{ 1, 2 \}$

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

LHS

$B \cup C = \{ 0, 1, 2 \}$

$A \times (B \cup C) = \{ 2, 3 \} \times \{ 0, 1, 2 \}$

$= \{ (2, 0), (2, 1), (2, 2), (3, 0),$

$(3, 1), (3, 2) \}$ ...(1)

RHS

$A \times B = \{ 2, 3 \} \times \{ 0, 1 \}$

$= \{ (2, 0), (2, 1), (3, 0), (3, 1) \}$

$A \times C = \{ 2, 3 \} \times \{ 1, 2 \}$

$= \{ (2, 1), (2, 2), (3, 1), (3, 2) \}$

$(A \times B) \cup (A \times C) = \{(2, 0), (2, 1), (2, 2), \}$

$(3, 0), (3, 1), (3, 2)$ ...(2)

From (1) & (2)

$A \times (B \cup C) = (A \times B) \cup (A \times C)$ is verified

(ii) $A \times B \cap C) = (A \times B) \cap (A \times C)$

LHS

$B \cap C) = \{ 1 \}$

$A \times (B \cap C) = \{ 2, 3 \} \times \{ 1 \}$

$= \{ (2, 1), (3, 1) \}$ ...(1)

$A \times B = \{ (2, 0), (2, 1), (3, 0), (3, 1) \}$

$A \times C = \{ (2, 1), (2, 2), (3, 1), (3, 2) \}$

$\therefore \{ (A \times B) \cap (A \times C) = (2, 1), (3, 1) \}$ ...(2)

From (1) & (2)

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ is verified.

Type: II Given $A \times B$ Find the sets A and B QNo.3, Example 1.2

3. If $B \times A = \{ (-2, 3), (-2, 4), (0, 3), (0, 4), (3, 3), (3, 4) \}$ find A and B

We have

$A = \{$ Set of all first co – ordinates of elements of $(A \times B) \}$

$\therefore A = \{ -2, 0, 3 \}$

$B = \{$ Set of all second co – ordinate of elements of $A \times B \}$

$B = \{ 3, 4 \}$

Example 1.2

If $A \times B = \{ (3, 2), (3, 4), (5, 2), (5, 4) \}$ then find A and B

We have

$A = \{$ Set of all first co – ordinates of elements of $A \times B \}$

$\therefore A = \{ 3, 5 \}$

$B = \{$ Set of all second co – ordinates of elements of $A \times B \}$

$B = \{ 2, 4 \}$

Exercise 1.2

KEY POINTS

Relation

Let A and B be any two non-empty sets. A **Relation** R from A to B is a subset of $A \times B$ satisfying some specified conditions.

If $x \in A$ is related to $y \in B$ through A, then we write it has xRy. xRy if and only if $(x, y) \in R$.

Domain

The domain of the Relation

$R = \{ x \in A \mid xRy$, for some $y \in B \}$

Co-domain

The co-domain of the relation R is B.

Range

The range of the relation $R = \{ y \in B \mid xRy$, for some $x \in A \}$ Range is a subset of co-domain.

Note

(i) A relation may be represented algebraically either by the roster method or by the set builder method.

(ii) An arrow diagram is a visual representation of a relation.

(iii) A relation which contains no element is called a "Null relation".

(iv) If $n(A) = p$, $n(B) = q$ then the total number of relations that exist between A and B is 2^{pq}.

Type: I Verify Relation or not
QNo: 1. (i) (ii) (iii) (iv). Example 1.4 (i) (ii) (iii)

1. Let $A = \{ 1, 2, 3, 7 \}$ and $B = \{ 3, 0, -1, 7 \}$ which of the following are relation from A to B?

Here

$A \times B = \{ 1, 2, 3, 7 \} \times \{ 3, 0, -1, 7 \}$

$= \{ (1, 3), (1, 0), (1, -1), (1, 7), (2, 3), (2, 0), (2, -1), (2, 7), (3, 3), (3, 0)(3, -1), (3, 7), (7, 3), (7, 0), (7, -1), (7, 7) \}$

(i) $R_1 = \{ (2, 1), (7, 1) \}$

Here $R_1 \not\subseteq A \times B$. So R_1 is not a Relation from A to B.

(ii) $R_2 = \{ (-1, 1) \}$

Here $R_2 \not\subseteq A \times B$. So R_2 is not a Relation from A to B.

(iii) $R_3 = \{ (2, -1), (7, 7), (1, 3) \}$

Here $R_3 \subseteq A \times B$. So R_3 is a Relation from A to B.

(iv) $R_4 = \{ (7, -1), (0, 3), (3, 3), (0, 7) \}$

Here $R_4 \not\subseteq A \times B$. So R_4 is not a Relation from A to B.

Example 1.4

Let $A = \{ 3, 4, 7, 8 \}$ ***and*** $B = \{ 1, 7, 10 \}$. ***which of the following sets are relations from A to B?***

Here $A \times B = \{ (3, 1), (3, 7), (3, 10), (4, 1), (4, 7), (4, 10), (7, 1), (7, 7), (7, 10), (8, 1), (8, 7), (8, 10) \}$

(i) $R_1 = \{ (3, 7), (4, 7), (7, 10), (8, 1) \}$

Here $R_1 \subseteq A \times B$. So R_1 is a Relation from A to B.

(ii) $R_2 = \{ (3, 1), (4, 12) \}$

Here $R_2 \not\subseteq A \times B$. So R_2 is not a Relation from A to B. Note $(4, 12) \notin A \times B$

(iii) $R_3 = \{ (3, 7), (4, 10), (7, 7), (7, 8), (8, 11), (8, 7), (8, 10) \}$

Here $R_3 \not\subseteq A \times B$. So R_3 is not a Relation from A to B. Note $(7,8) \notin A \times B$.

Type: II Find Domain and Range of a Relation
QNo: 2,3

2. Let $A = \{ 1, 2, 3, 4, \ldots 45 \}$ and R be the relation defined as "is square of" on A. Write R as a subset of $A \times A$. Also, find the domain and range of R.

$$A = \{ 1, 2, 3, 4 \ldots 45 \}$$

Relation - is square of

- $R = \{ (1, 1), (2, 4), (3, 9), (4, 16), (5, 25), (6, 36) \}$

Here $R \subseteq A \times A$

- Domain of $R = \{ 1, 2, 3, 4, 5, 6 \}$
- Range of $R = \{ 1, 4, 9, 16, 25, 36 \}$

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3. A Relation R is given by the set $\{ (x, y) \mid y = x + 3 \}, x \in \{ 0, 1, 2, 3, 4, 5 \}$. Determine its domain and Range.

$x = \{ 0, 1, 2, 3, 4, 5 \}$

$y = x + 3$

$= 0 + 3$	$1 + 3$	$2 + 3$	$3 + 3$	$4 + 3$	$5 + 3$
$= 3$	$= 4$	$= 5$	$= 6$	$= 7$	$= 8$

$\therefore y = \{ 3, 4, 5, 6, 7, 8 \}$

$R = \{ (0, 3), (1, 4), (2, 5), (3, 6), (4, 7), (5, 8) \}$

$\therefore$ Domain of $R = \{ 0, 1, 2, 3, 4, 5 \}$

Range of $R = \{ 3, 4, 5, 6, 7, 8 \}$

Type: III Representation of a Relation
QNo: 4.(i)(ii), 5, Example 1.5

4. Represent each of the given relations by (a) an arrow diagram (b) a graph and (c) a set in roster form, wherever possible.

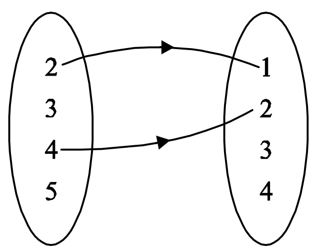
(i) $(x + y) \mid x = 2y, x \in \{ 2, 3, 4, 5 \} y \in \{ 1, 2, 3, 4 \}$

$x = \{ 2, 3, 4, 5 \}; \{ y = 1, 2, 3, 4 \}$

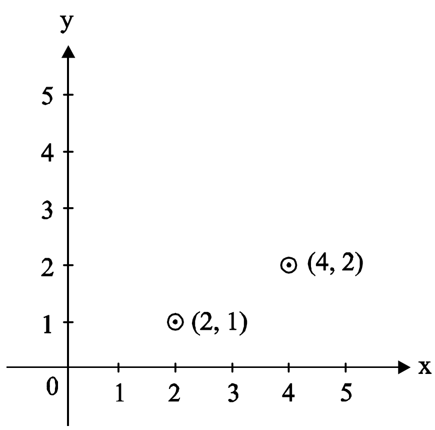
$$R \Rightarrow \quad \begin{array}{rl|rl} x &= 2y & x &= 2y \\ &= 2\,(1) & &= 2\,(2) \\ &= 2 & &= 4 \end{array}$$

$$R = \{\ (1, 2), (2, 4)\ \}$$

(a) an arrow diagram



(b) a graph



(c) a set in roster form

$R = \{\ (1, 2), (2, 4)\ \}$

(ii) $\{\ (x, y) \mid y = x + 3, x, y$ **are natural numbers < 10}**

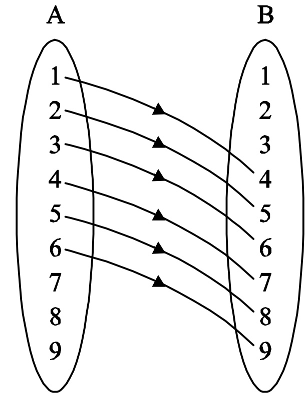
$x = \{\ 1, 2, 3, 4, 5, 6, 7, 8, 9\ \}$

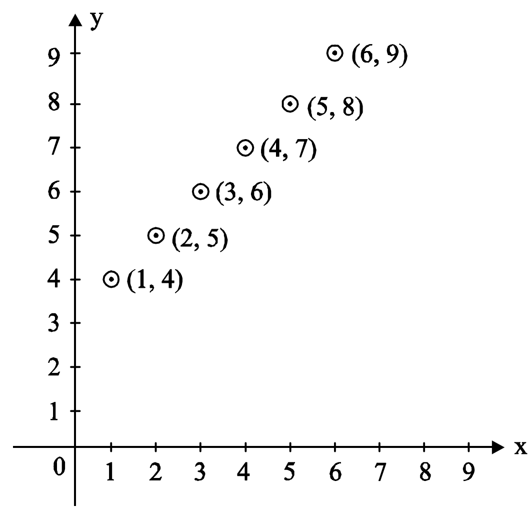
$y = \{\ 1, 2, 3, 4, 5, 6, 7, 8, 9\ \}$

$R = \{\ (x, y) \mid y = x + 3\ \}$

$R = \{\ (1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\ \}$

(a) an arrow diagram





(c) a set in roster form

$R = \{\ (1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\ \}$

5. A company has four categories of employees given by Assistants (A) Clerks (C) Managers (M) and an Executive officer (E). The company provide Rs.10,000, Rs.25,000, Rs.50,000 and Rs.1,00,000 as salaries to the people who work in categories A, C, M and E respectively. If A_1, A_2, A_3, A_4 and A_5 were Assistants, C_1, C_2, C_3, C_4 were clerks; M_1, M_2, M_3 were managers and E_1, E_2 were Executive Officers and if the relation R is defined by xRy, where x is the salary given to person y, express the relation R through an ordered pair and an arrow diagram.

Assistants $(A) \rightarrow A_1, A_2, A_3, A_4, A_5$

Clerks $(C) \rightarrow C_1, C_2, C_3, C_4$

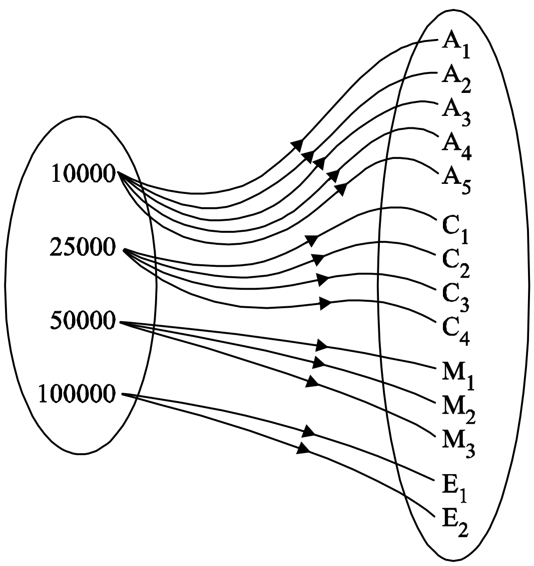
Managers $(M) \rightarrow M_1, M_2, M_3$

Executive officer $(E) \rightarrow E_1, E_2$

(a) an ordered pair

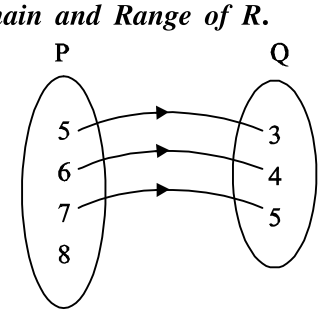
$R = \{ (10,000, A_1), (10000, A_2), (10000, A_3), (10000, A_4), (10000, A_5), (25000, C_1), (25000, C_2), (25000, C_3), (25000, C_4) (50000, M_1), (50000, M_2), (50000, M_3) (100000, E_1), (100000, E_2) \}$

(b) arrow diagram



Example 1.5

The arrow diagram shows a relationship between the sets P and Q write the relation in (i) Set builder form (ii) Roster form (iii) What is the domain and Range of R.



(i) Set builder form of R

$R = \{ (x, y) \mid y = x - 2, x \in P, y \in Q \}$

(ii) Roster form of R

$R = \{ (5, 3), (6, 4), (7, 5) \}$

Domain of $R = \{ 5, 6, 7 \}$

Range of $R = \{ 3, 4, 5 \}$

Exercise 1.3

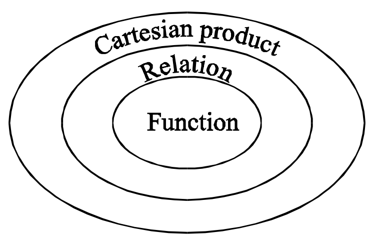
KEY POINTS

1. Function

A relation f between two non empty sets X and Y is called a function from X to Y, if for each $x \in X$ there exists one $y \in Y$, such that $(x\ y) \in f$.

That is

$f = \{ (x, y) \text{ for all } x \in X, y \in Y \}$



Functions are subsets of relations, and relations are subsets of cartesian product.

A function is also called as a mapping or transformation.

Note

If $f: X \rightarrow Y$ is a function then

- The set X is called the domain of the function f and the set Y is called its co-domain.
- If $f(a) = b$, then b 'image' of a under f and a is called 'pre-image of b,
- The set of all images of the elements of X under f is called the 'range' of f.
- $f: X \rightarrow Y$ is a function only if
  (i) every element in the domain of f has an

image.
(ii) the image is unique.

- If A and B are finite sets such that $n(A) = p, n(B) = q$ then the total number of functions that exist between A and B is q^p.
- In this chapter we always consider f to be a real valued function.
- Describing domain of a function.

(i) Let $f(x) = \frac{1}{x+1}$. If $x = -1$ then $f(-1)$ is not defined. Hence f is defined for all real numbers except at $x = -1$. So domain of f is $R - \{-1\}$.

(ii) Let $f(x) = \frac{1}{x^2 - 5x + 6}$; if $x = 2, 3$ then $f(2)$ and $f(3)$ are not defined. Hence f is defined for all real numbers except at $x = 2$ and 3. So domain of $f = R - \{2, 3\}$.

Type: I Verify function or not
QNo: 1,2, Example 1.8

1. Let $f = \{(x, y) \mid x, y \in N \text{ and } y = 2x\}$ be a relation on N. Find the domain co-domain and Range. Is this relation a function.

Here $X = \{1, 2, 3, 4 \dots\}$

$Y = \{2, 4, 6, 8 \dots\}$ Since Given $y = 2x$

$\therefore R = \{(1, 2), (2, 4), (3, 6), (4, 8), \dots\}$

- Domain $= \{1, 2, 3, 4 \dots\}$
- Co-domain $= \{1, 2, 3, 4, \dots\}$
- Range $= \{2, 4, 6, 8 \dots\}$

This Relation is a function as each element of X has an unique image in Y.

2. Let $X = \{3, 4, 6, 8\}$. Determine whether the Relation $R = \{(x, f(x)) \mid x \in X, f(x) = x^2 + 1\}$ is a function from X to N.

$X = \{3, 4, 6, 8\}$

$y = f(x) = x^2 + 1$

$y = \{3^2 + 1, 4^2 + 1, 6^2 + 1, 8^2 + 1\}$

$y = \{10, 17, 37, 65\} \in N$

So It is a function from X to N.

Example 1.8

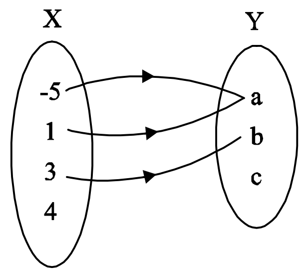
If $X = \{-5, 1, 3, 4\}$ and $Y = \{a, b, c\}$, then which of the following relations are functions from X to Y?

***(i)* $R_1 = \{(-5, a), (1, a), (3, b)\}$**

***(ii)* $R_2 = \{(-5, b), (1, b), (3, a), (4, c)\}$**

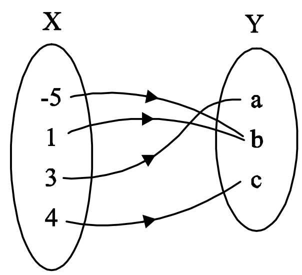
***(iii)* $R_3 = \{(-5, a), (1, a), (3, b), (4, c), (1, b)\}$**

(i) $R_1 = \{(-5, a), (1, a), (3, b)\}$



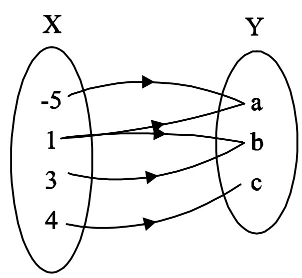
- It is not a function.
- **Reason:** In X 4 has no image in Y.

(ii) $R_2 = \{(-5, b), (1, b), (3, a), (4, c)\}$



- It is a function.
- **Reason:** each element in X has an unique image in Y.

(iii) $R_3 = \{(-5, a), (1, a), (3, b), (4, c), (1, b)\}$



- It is not a function.
- **Reason:** In X an element 1 has two images a, b in Y.

Type: II Find the value of $f(x)$

QNo: 3, 4, 5, 6, 8 Example 1.9

3. Given the function $f : x \rightarrow x^2 - 5x + 6$. Evaluate (i) $f(-1)$ (ii) $f(2a)$ (iii) $f(2)$ (iv) $f(x-1)$

Given

$f(x) = x^2 - 5x + 6$

(i) $f(-1) = (-1)^2 - 5(-1) + 6$

$= 1 + 5 + 6$

$= 12$

(ii) $f(2a) = (2a)^2 - 5(2a) + 6$

$= 4a^2 - 10a + 6$

(iii) $f(2) = (2)^2 - 5(2) + 6$

$= 4 - 10 + 6$

$= 0$

(iv) $f(x-1) = (x-1)^2 - 5(x-1) + 6$

$= x^2 - 2x + 1 - 5x + 5 + 6$

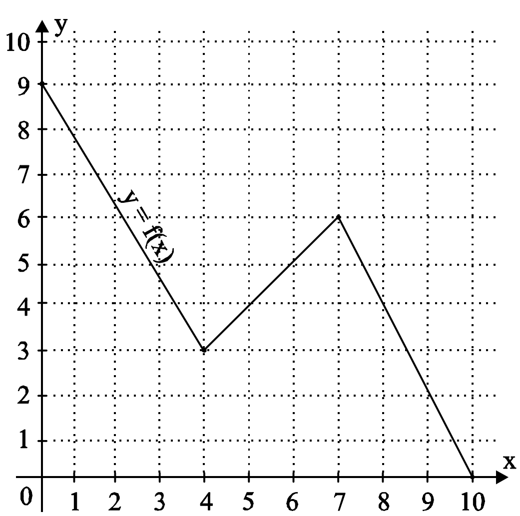
$= x^2 - 7x + 12$

---

4. A graph representing the function $f(x)$ is given in figure, it is clear that $f(9) = 2$.

(i) Find the following values of the function

(a) $f(0)$ (b) $f(7)$ (c) $f(2)$ (d) $f(10)$



(ii) For what value of x is $f(x) = 1$

(iii) Describe the following (i) Domain (ii) Range

(iv) What is the image of 6 under f?

From the graph, we get

(i) (a) $f(0) = 9$ (b) $f(7) = 6$

(c) $f(2) = 6$ (d) $f(10) = 0$

(ii) $f(x) = 1$ if $x = 9.5$

(iii) (i) Domain $= \{ x \mid 0 \le x \le 10, x \in R \}$

(ii) Range $= \{ x \mid 0 \le x \le 9, x \in R \}$

(iv) Image of 6 is 5

---

5. Let $f(x) = 2x + 5$. If $x \neq 0$ then find $\dfrac{f(x+2) - f(2)}{x}$

Given $f(x) = 2x + 5$

$f(x+2) = 2(x+2) + 5$

$= 2x + 4 + 5$

$= 2x + 9$

$f(2) = 2(2) + 5$

$= 4 + 5$

$= 9$

Now $\dfrac{f(x+2) - f(z)}{x} = \dfrac{2x + 9 - 9}{x}$

$= \dfrac{2x}{x}$

$= 2$

---

6. A function f is defined by $f(x) = 2x - 3$

(i) find $\dfrac{f(0) + f(1)}{2}$

(ii) find x such that $f(x) = 0$

(iii) find x such that $f(x) = x$

(iv) find x such that $f(x) = f(1-x)$

Given $f(x) = 2x - 3$

(i) $\dfrac{f(0) + f(1)}{2} = \dfrac{[2(0) - 3] + [2(1) - 3]}{2}$

Type: II Representation of a function

QNo: 2,3, Example 1.11

2. **Let $f: A \to B$ be a function defined by $f(x) = x/2 - 1$ where $A = \{ 2, 4, 6, 10, 12 \}$ $B = \{ 0, 1, 2, 4, 5, 9 \}$. Represent f by (i) set of ordered pairs (ii) a table (iii) an arrow diagram (iv) a graph**

$A = \{ 2, 4, 6, 10, 12 \}$; $B = \{ 0, 1, 2, 4, 5, 9 \}$

$$\boxed{f(x) = \frac{x}{2} - 1}$$

$$f(2) = \frac{2}{2} - 1 = 1 - 1 = 0$$

$$f(4) = \frac{4}{2} - 1 = 2 - 1 = 1$$

$$f(6) = \frac{6}{2} - 1 = 3 - 1 = 2$$

$$f(10) = \frac{10}{2} - 1 = 5 - 1 = 4$$

$$f(12) = \frac{12}{2} - 1 = 6 - 1 = 5$$

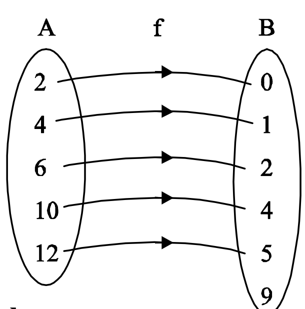
(i) Set of ordered pairs

$f = \{ (2, 0), (4, 1), (6, 2), (10, 4), (12, 5) \}$

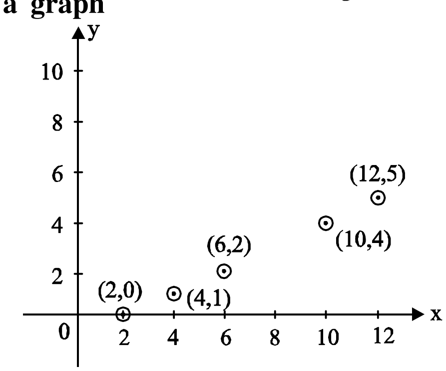
(ii) a table

x	2	4	6	10	12
y	0	1	2	4	5

(iii) an arrow diagram



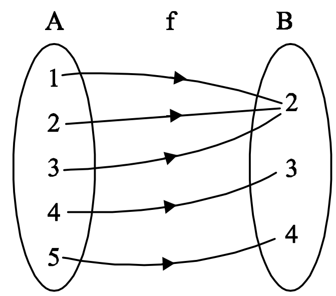
(iv) a graph



3. **Represent the function $f = \{ (1, 2), (2, 2), (3, 2), (4, 3), (5, 4) \}$ through (i) an arrow diagram (ii) a table form (iii) a graph**

$f = \{ (1, 2), (2, 2), (3, 2), (4, 3), (5, 4) \}$

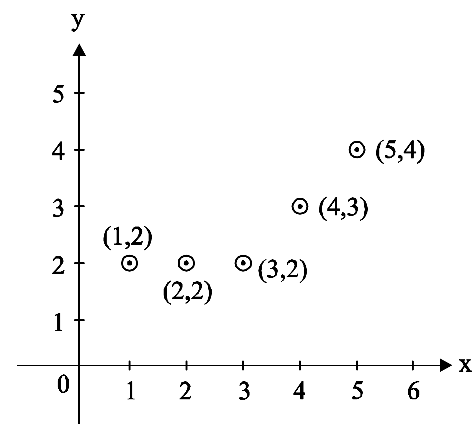
(i) an arrow diagram



(ii) a table

x	1	2	3	4	5
y	2	2	2	3	4

(iii) a graph



Example 1.11

Let $A = \{ 1, 2, 3, 4 \}$ and $B = \{ 2, 5, 8, 11, 14 \}$ be two sets, let $f: A \to B$ be a function given by $f(x) = 3x - 1$. Represent this function.

(i) by arrow diagram (ii) in a table form

(iii) as a set of ordered pairs

(iv) in a graphical form

$A = \{ 1, 2, 3, 4 \}$; $B = \{ 2, 5, 8, 11, 14 \}$

$$\boxed{f(x) = 3x - 1}$$

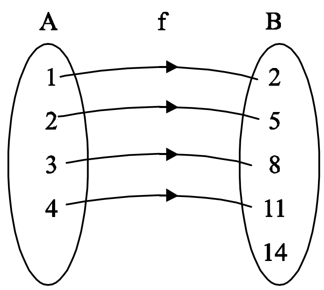
$f(1) = 3(1) - 1 = 3 - 1 = 2$

$f(2) = 3(2) - 1 = 6 - 1 = 5$

$f(3) = 3(3) - 1 = 9 - 1 = 8$

$f(4) = 3(4) - 1 = 12 - 1 = 11$

(i) Arrow diagram



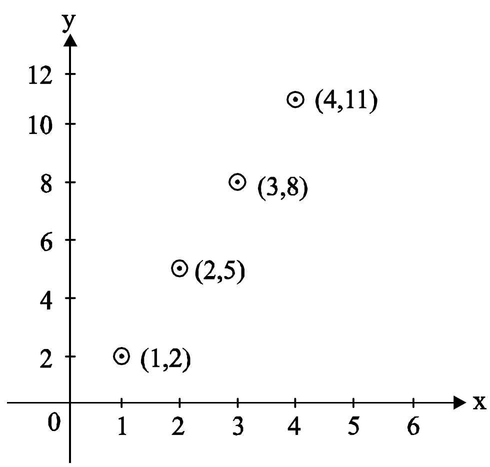
(ii) A table

x	1	2	3	4
y	2	5	8	11

(iii) Set of ordered pairs

$f = \{ (1, 2), (2, 5), (3, 8), (4, 11) \}$

(iv) Graph



Type: III Problems based on type of functions QNo: 4, 5, 6, 7 Example 1.13, 1.15, 1.14, 1.17

QNo: 8, Example 1.16, 11, 1.12

4. Show that the function $f: N \to N$ defined by $f(x) = 2x - 1$ is one-one but not onto.

$f(x) = 2x - 1$ and $f: N \to N$

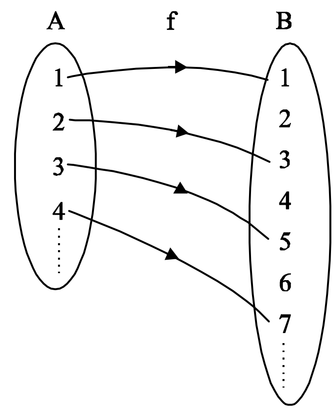
$\therefore$ Domain $= N =$ co-domain

$f(1) = 2(1) - 1 = 2 - 1 = 1$

$f(2) = 2(2) - 1 = 4 - 1 = 3$

$f(3) = 2(3) - 1 = 6 - 1 = 5$

$f(4) = 2(4) - 1 = 8 - 1 = 7$



Here In Domain A each element have different image in co-domain B. So it is one-one function.

Here Range $\neq$ Co-domain

So it is not a onto function

5. Show that the function $f: N \to N$ defined by $f(m) = m^2 + m + 3$ is one-one function.

$f(m) = m^2 + m + 3$; $f: N \to N$

Put: $m = 1, 2, 3 \ldots$

Domain $= N$

Co-domain $= N$

$$f(m) = m^2 + m + 3$$

$$f(1) = 1 + 1 + 3 = 5$$

$$f(2) = 2^2 + 2 + 3 = 9$$

$$f(3) = 3^2 + 3 + 3 = 15$$

$$f(4) = 4^2 + 4 + 3 = 23$$

$\therefore$ Here In domain A, each element have different image in co-domain B.

$\therefore$ It is one-one function.

6. Let $A = \{1, 2, 3, 4\}$ and $B = N$. Let $f: A \to B$ be defined by $f(x) = x^3$ than
(i) find the range of f
(ii) identify the type of function

$A = \{1, 2, 3, 4\}$; $B = N$

$f(x) = x^3$

$f(1) = 1^3 = 1$

$f(2) = 2^3 = 8$

$f(3) = 3^3 = 27$

$f(4) = 4^3 = 64$

(i) Range of $f = \{1, 8, 27, 64\}$

(ii) Type of function: **one-one and into function** since each element in domain A have different image in co-domain B.

---

7. In each of the following cases state whether the function is bijective or not. Justify your answer.

(i) $f: R \to R$ defined by $f(x) = 2x + 1$

(ii) $f: R \to R$ defined by $f(x) = 3 - 4x^2$

(i) $f: R \to R$ defined by $f(x) = 2x + 1$

$f(-2) = 2(-2) + 1 = -4 + 1 = -3$

$f(-1) = 2(-1) + 1 = -2 + 1 = -1$

$f(0) = 2(0) + 1 = 0 + 1 = 1$

$f(1) = 2(1) + 1 = 2 + 1 = 3$

$f(2) = 2(2) + 1 = 4 + 1 = 5$

Here all elements in Domain have different images and all elements in co-domain have pre-images in Domain. So it is one-one and on to function (bijective function).

(ii) $f: R \to R$ defined by $f(x) = 3 - 4x^2$

$f(x) = 3 - 4x^2$

$f(-2) = 3 - 4(-2)^2 = 3 - 16 = -13$

$f(-1) = 3 - 4(-1)^2 = 3 - 4 = -1$

$f(0) = 3 - 4(0)^2 = 3 - 0 = 3$

$f(1) = 3 - 4(1)^2 = 3 - 4 = -1$

$f(2) = 3 - 4(2)^2 = 3 - 16 = -13$

Here In domain element $1, -1$ and $2, -2$ has same image so it is not one-one function. Range is a subset of co-domain so it is not onto function.

$\therefore f(x)$ is not a bijective function.

Example 1.13

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function and $f = \{(1, 4), (2, 5), (3, 6)\}$ be a junction from A to B. Show that f is one-one but not on to function.

$A = \{1, 2, 3\}$; $B = \{4, 5, 6, 7\}$

$f = \{(1, 4), (2, 5), (3, 6)\}$ then f is a function from A to B and for different elements in A, there are different images in B. Hence f is one-one function.

In co-domain B element 7 does not have pre-image in the domain. Hence f is not onto.

$\therefore f$ is one-one but not an onto function.

Example 1.15

Let f be a function $f: N \to N$ be defined by $f(x) = 3x + 2, x \in N$.
(i) Find the images of 1, 2, 3
(ii) Find pre-images of 29, 53
(iii) Identify the type of function

The function $f: N \to N$ is defined by $f(x) = 3x + 2$

(i) Images of 1, 2, 3

$f(1) = 3(1) + 2 = 5$

$f(2) = 3(2) + 2 = 8$

$f(3) = 3(3) + 2 = 11$

$\therefore$ image of 1 is 5
2 is 8
3 is 11

(ii) Pre-images of 29, 53

$$f(x) = 3x + 2$$
$$29 = 3x + 2$$
$$29 - 2 = 3x$$
$$27 = 3x$$
$$\frac{27}{3} = x$$
$$9 = x$$

$$53 = 3x + 2$$
$$53 - 2 = 3x$$
$$51 = 3x$$
$$\frac{51}{3} = x$$
$$17 = x$$

$\therefore$ Pre-image of 29 is 9

Pre-image of 53 is 17

(iii) Type of function

Since different elements of N have different images in the co-domain, the function f is one-one function.

Here co-domain $= N = \{ 1, 2, 3 \ldots \}$

Range $= \{ 5, 8, 11 \}$

$\therefore$ It is not an onto function.

It is into function.

$\therefore$ Thus f is one-one and into function.

Example 1.14

If $A = \{ -2, -1, 0, 1, 2 \}$ and $f: A \to B$ is an onto function defined by $f(x) = x^2 + x + 1$ then find B.

$$A = \{ -2, -1, 0, 1, 2 \}$$

$$f(x) = x^2 + x + 1$$
$$= (-2)^2 + (-2) + 1$$
$$= 4 - 2 + 1$$
$$= 3$$

$$= (-1)^2 + (-1) + 1$$
$$= 1 - 1 + 1$$
$$= 1$$

$$= (0)^2 + (0) + 1$$
$$= 0 + 0 + 1$$
$$= 1$$

$$= (1)^2 + (1) + 1$$
$$= 1 + 1 + 1$$
$$= 3$$

$$= (2)^2 + (2) + 1$$
$$= 4 + 2 + 1$$
$$= 7$$

Since f is an onto function range $f = B =$ co-domain of f

$\therefore B = \{ 1, 3, 7 \}$

8. Let $A = \{ -1, 1 \}$ and $B = \{ 0, 2 \}$. If the function $f: A \to B$ defined by $f(x) = ax$ is an onto function? Find a and b

$$A = \{ -1, 1 \}; B = \{ 0, 2 \}$$
$$f(x) = ax + b$$

Given $f(x)$ is an onto function

$\therefore$ Range = co-domain

$$f(-1) = 0$$
$$a(-1) + b = 0$$
$$-a + b = 0 \quad \ldots(1)$$

$$f(1) = 2$$
$$a(1) + b = 2$$
$$a + b = 2 \quad \ldots(2)$$

$$-a + b = 0$$
$$a + b = 2$$
$$2b = 2$$
$$\boxed{b = 1}$$

Put $b = 1$ in (1)

$$-a + 1 = 0$$
$$\boxed{a = 1}$$

Example 1.17

Let f be a function from R to R defined by $f(x) = 3x - 5$. Find the values of a and b given that $(a, 4)$ and $(1, b)$ belong to f.

$$f(x) = 3x - 5$$

$(a, 4)$ means image of a is 4

$$f(a) = 4$$
$$3a - 5 = 4$$
$$3a = 4 + 5$$
$$a = \frac{9}{3}$$
$$\boxed{a = 3}$$

$(1, b)$ means pre-image of b is 1

$$b = 3(1) - 5$$
$$b = 3 - 5$$
$$\boxed{b = -2}$$

Example 1.16

Forensic scientists can determine the height (in cms) of a person based on the length of their thigh bone. They usually do so using the function.

$h(b) = 2.47b + 54.10$ *where b is the length of the thigh bone.*

(i) Check if the function h is one-one.

(ii) Also find the height of a person if the length of his thigh bone is 50 cm.

(iii) Find the length of the thigh hone if the height of a person is 147.96 cm.

(i) To check if h is one-one

Assume that $h(b_1) = h(b_2)$

$$2.47\, b_1 + 54.10 = 2.47\, b_2 + 54.10$$

$$2.47\, b_1 = 2.47\, b_2$$

$$b_1 = b_2$$

Thus $h(b_1) = h(b_2) \Rightarrow b_1 = b_2$. So the function h is one-one.

(ii) Given $b = 50$

$$\therefore h(50) = 2.47(50) + 54.1$$

$$= 123.5 + 54.1$$

$$= 177.6 \text{ cms}$$

(iii) $h(b) = 147.96, b = ?$

$$2.47b + 54.10 = 147.96$$

$$2.47b = 147.96 - 54.10$$

$$2.47b = 93.86$$

$$b = \frac{93.86}{2.47}$$

$$b = 38 \text{ cm}$$

11. The distance S an object travels under the influence of gravity in time t seconds is given by $S(t) = \frac{1}{2}gt^2 + at + b$. Where ($g$ is the acceleration due to gravity) a, b are constant. Check if the function $S(t)$ is one-one.

$$S(t) = \frac{1}{2}gt^2 + at + b$$

Put $t = 1, 2, 3 \ldots$

$$S(1) = \frac{1}{2}g + a + b$$

$$f(2) = \frac{1}{2}g(2)^2 + a(2) + b = 2g + 2a + b$$

$$f(3) = \frac{1}{2}g(3)^2 + a(3) + b$$

$$= \frac{9g}{2} + 3a + b$$

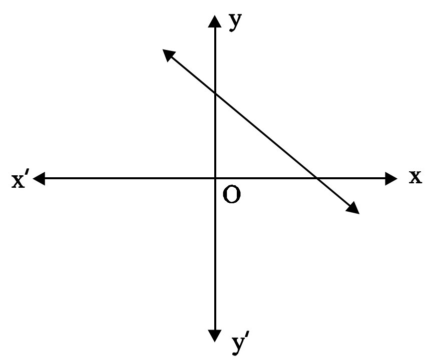
It is **one-one** function because

For different values of t we get different images.

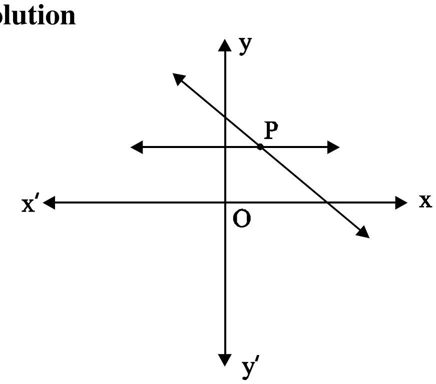
Example 1.12

Using horizontal line test, determine which of the following functions are one-one.

(a)

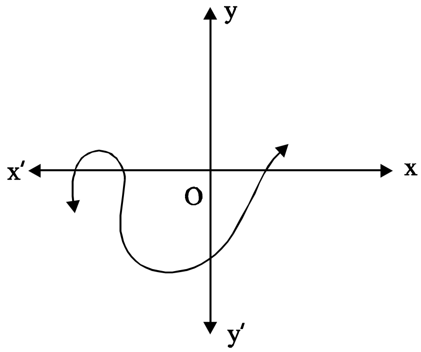


✍ **Solution**

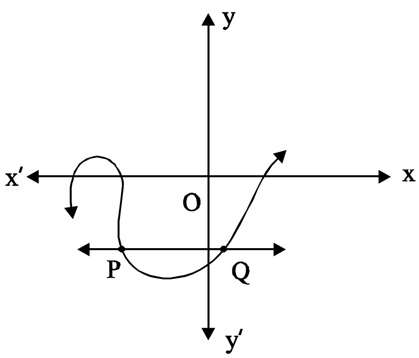


- It is one-one function.
- **Reason:** Horizontal line meet the curve in only one point *P*.

(b)

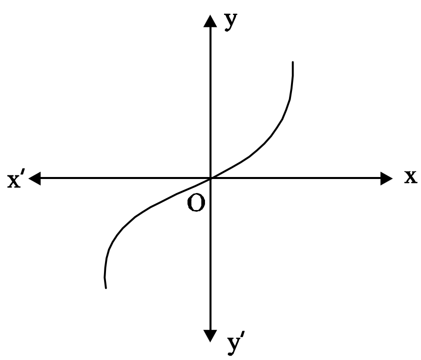


✍ **Solution**

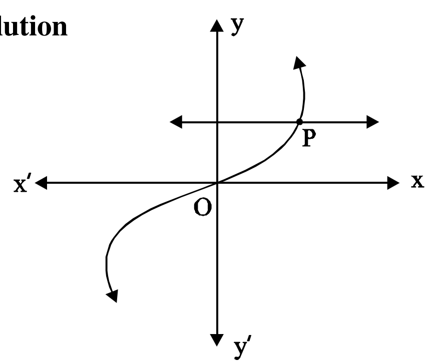


- It is not one-one function.
- **Reason:** Horizontal line meet the curve at two points *P* and *Q*.

(c)



✍ **Solution**



- It is one-one function.
- **Reason:** Horizontal line meet the curve in only one point *P*.

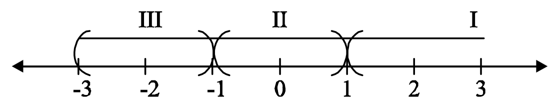
Type: IV $f(x)$ based problems

QNo: 9, 10 Example 1.19, 1.18, QNo: 12

9. If the function f is defined by

$$f(x)=\begin{cases} x+2 & \text{if } x>1 \\ 2 & \text{if } -1\le x\le 1 \\ x-1 & \text{if } -3<x<-1 \end{cases}$$

find the values of (i) $f(3)$ (ii) $f(0)$ (iii) $f(-1.5)$ (iv) $f(2)+f(-2)$



(i) $f(3)$

3 lies in I interval

$$\therefore f(3)=3+2=5$$

(ii) $f(0)$

0 lies in II interval

$$f(0)=2$$

(iii) $f(-1,5)$

-1.5 lies in III - interval

$$\therefore f(-1.5)=-1.5-1$$
$$=-2.5$$

(iv) $f(2)+f(-2)$

2 lies in I interval

$$\therefore f(2)=2+2=4$$

-2 lies in III interval

$$f(-2)=-2-1$$
$$=-3$$
$$\therefore f(2)+f(-2)=4-3$$
$$=1$$

10. A function $f: [-5, 9] \rightarrow R$ is defined as follows.

$$f(x) = \begin{cases} 6x + 1 & \text{if } -5 \le x < 2 \\ 5x^2 - 1 & \text{if } 2 \le x < 6 \\ 3x - 4 & \text{if } 6 \le x \le 9 \end{cases}$$

Find

(i) $f(-3) + f(2)$ **(ii)** $f(7) - f(1)$

(iii) $2f(4) + f(8)$ **(iv)** $\dfrac{2f(-2) - f(6)}{f(4) + f(-2)}$

- $f(x) = 6x + 1$, $x = -5, -4, \mathbf{-3}, \mathbf{-2}, -1, 0, \mathbf{1}$

$f(-3) = 6(-3) + 1$
$= -18 + 1$
$= -17$

$f(-2) = 6(-2) + 1$
$= -12 + 1$
$= -11$

$f(1) = 6(1) + 1$
$= 6 + 1$
$= 7$

- $f(x) = 5x^2 - 1$; $x = \mathbf{2}, 3, \mathbf{4}, 5$

$f(2) = 5(2)^2 - 1$
$= 20 - 1$
$= 19$

$f(4) = 5(4)^2 - 1$
$= 80 - 1$
$= 79$

- $f(x) = 3x - 4$; $x = \mathbf{6, 7, 8,} 9$

$f(6) = 3(6) - 4$
$= 18 - 4$
$= 14$

$f(7) = 3(7) - 4$
$= 21 - 4$
$= 17$

$f(8) = 3(8) - 4$
$= 24 - 4$
$= 20$

(i) $f(-3) + f(2) = -17 + 19 = 2$

(ii) $f(7) - f(1) = 17 - 7 = 10$

(iii) $2f(4) + f(8) = 2(79) + 20$
$= 158 + 20$
$= 178$

(iv) $\dfrac{2f(-2) - f(6)}{f(4) + f(-2)} = \dfrac{2(-11) - 14}{79 + (-11)}$
$= \dfrac{-22 - 14}{79 - 11}$
$= \dfrac{-36}{68}$
$= \dfrac{-9}{17}$

Example 1.19

If the function $f: R \rightarrow R$ defined by

$$f(x) = \begin{cases} 2x + 7, & x < -2 \\ x^2 - 2, & -2 \le x < 3 \\ 3x - 2, & x \ge 3 \end{cases}$$

then find the values of (i) $f(4)$ (ii) $f(-2)$ (iii) $f(4) + 2f(1)$ (iv) $\dfrac{f(1) - 3f(4)}{f(-3)}$

- $f(x) = 2x + 7$; $x = \mathbf{-3}, -4, -5$

$f(-3) = 2(-3) + 7$
$= -6 + 7$
$= 1$

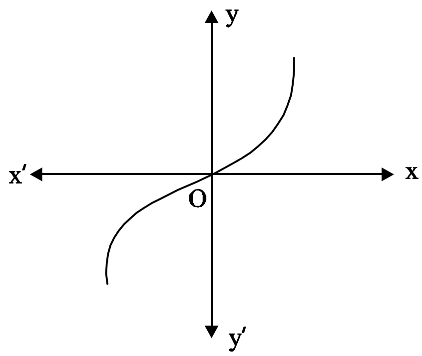
- $f(x) = x^2 - 2$; $x = \mathbf{-2}, -1, 0, \mathbf{1}, 2$

$f(-2) = (-2)^2 - 2$
$= 4 - 2$
$= 2$

$f(1) = (1)^2 - 2$
$= 1 - 2$
$= -1$

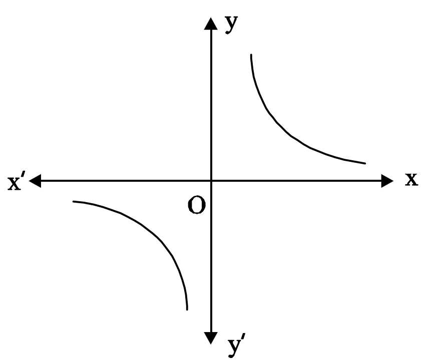
4. Cubic functions

A function $f: R \to R$ defined by $f(x) = ax^3 + bx^2 + cx + d, (a \neq 0)$ is called a cubic function.



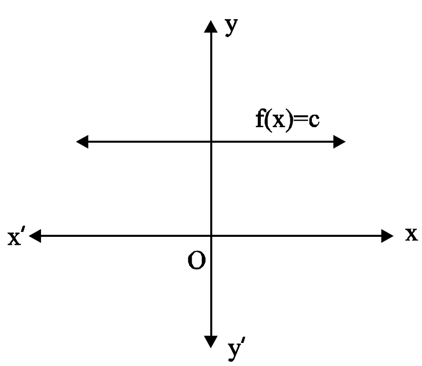
5. Reciprocal functions

A function $f: R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ is called a reciprocal function.



6. Constant function

A function $f: R \to R$ defined by $f(x) = c$ for all $x \in R$ is called a constant function.



Type: I [Composition of functions based problems]

QNo: 1. (i) (ii) (iii) (iv) 3, 5, 6, Example 1.20, 1.21 2. (i) (ii) 4. (i) (ii), 7, Example 1.22, 1.23 8. (i) (ii) (iii), Example 1.24 1.25

1. Using the functions f and g given below, find fog and gof. Check whether $fog = gof$.

(i) $f(x) = x - 6;\ g(x) = x^2$

$$(fog)\,x = f[g(x)] = f(x^2) = x^2 - 6$$

$$(gof)\,x = g[f(x)] = g[x-6] = (x-6)^2 = x^2 - 12x + 36$$

Here $fog \neq gof$

(ii) $f(x) = \frac{2}{x};\ g(x) = 2x^2 - 1$

$$(fog)\,x = f[g(x)] = f(2x^2 - 1) = \frac{2}{2x^2 - 1}$$

$$(gof)\,x = g[f(x)] = g\left(\frac{2}{x}\right) = 2\left(\frac{2}{x}\right)^2 - 1 = 2\left(\frac{4}{x^2}\right) - 1 = \frac{8 - x^2}{x^2}$$

Here $fog \neq gof$

(iii) $f(x) = \frac{x+6}{3},\ g(x) = 3 - x$

$$(fog)\,x = f[g(x)] = f(3 - x) = \frac{3 - x + 6}{3} = \frac{9 - x}{3}$$

$$(gof)\,x = g[f(x)] = g\left[\frac{x+6}{3}\right] = 3 - \left(\frac{x+6}{3}\right) = \frac{9 - x - 6}{3} = \frac{3 - x}{3}$$

Here $fog \neq gof$

(iv) $\boldsymbol{f(x) = 3 + x;\ g(x) = x - 4}$

$(fog)\,x = f[g(x)]$
$= f(x - 4)$
$= 3 + x - 4$
$= x - 1$

$(gof) = g[f(x)]$
$= g[3 + x]$
$= 3 + x - 4$
$= x - 1$

Here $fog = gof$

(v) $\boldsymbol{f(x) = 4x^2 - 1;\ g(x) = 1 + x}$

$(fog)\,x = f[g(x)]$
$= f(1 + x)$
$= 4(1 + x^2) - 1$
$= 4(1 + 2x + x^2) - 1$
$= 4 + 8x + 4x^2 - 1$
$= 4x^2 + 8x + 3$

$(gof)\,x = g[f(x)]$
$= g[4x^2 - 1]$
$= 1 + 4x^2 - 1$
$= 4x^2$

Here $fog \neq gof$

3. **If** $\boldsymbol{f(x) = 2x - 1;\ g(x) = \frac{x+1}{2}}$**, show that** $\boldsymbol{fog = gof = x}$**.**

$(fog)\,x = f[g(x)]$
$= f\left(\frac{x+1}{2}\right)$
$= 2\left(\frac{x+1}{2}\right) - 1$
$= x + 1 - 1$
$= x$

$(gof)\,x = g[f(x)]$
$= g[2x - 1]$
$= \frac{2x - 1 + 1}{2}$
$= \frac{2x}{2}$
$= x$

$\therefore$ Hence proved.

5. **Let** $\boldsymbol{A, B, C \subseteq N}$ **and a function** $\boldsymbol{f: A \to B}$ **be defined by** $\boldsymbol{f(x) = 2x + 1}$ **and** $\boldsymbol{g: B \to C}$ **be defined by** $\boldsymbol{g(x) = x^2}$**. Find the range of** $\boldsymbol{fog}$ **and** $\boldsymbol{gof}$**.**

$f(x) = 2x + 1;\ g(x) = x^2$

$(fog)\,x = f[g(x)]$
$= f(x^2)$
$= 2x^2 + 1$

$\therefore$ Range $= \{\, y \mid y = 2x^2 + 1, x \in N \,\}$

$(gof)\,x = g[f(x)]$
$= g(2x + 1)$
$= (2x + 1)^2$

Range $= \{\, y \mid y = (2x + 1)^2, x \in N \,\}$

6. **Let** $\boldsymbol{f(x) = x^2 - 1}$**. Find**

(i) $\boldsymbol{fof}$ **(ii)** $\boldsymbol{fo\,fof}$

$f(x) = x^2 - 1$

(i) $(fof)\,x = f[f(x)]$
$= f(x^2 - 1)$
$= (x^2 - 1)^2 - 1$
$= x^4 - 2x^2 + 1 - 1$
$= x^4 - 2x^2$

(ii) $(fo\,fof)\,x = f[(fof)\,x]$
$= f[x^4 - 2x^2]$
$= (x^4 - 2x^2)^2 - 1$

Example 1.20

Find fog and gof when $\boldsymbol{f(x) = 2x + 1}$ ***and*** $\boldsymbol{g(x) = x^2 - 2}$

$(fog)\,x = f[g(x)]$
$= f(x^2 - 2)$
$= 2(x^2 - 2) + 1$
$= 2x^2 - 4 + 1$
$= 2x^2 - 3$

$(gof)\,x = g[f(x)]$
$= g(2x + 1)$
$= (2x + 1)^2 - 2$
$= 4x^2 + 4x + 1 - 2$
$= 4x^2 + 4x - 1$

Here $fog \neq gof$

Example 1.21

Represent the function $\boldsymbol{f(x) = \sqrt{2x^2 - 5x + 3}}$ ***as a composition of two functions.***

We get $f_2(x) = 2x^2 - 5x + 3$ and

$f_1(x) = \sqrt{x}$

Now

$f(x) = \sqrt{2x^2 - 5x + 3} = \sqrt{f_2(x)}$

$= f_1(f_2(x)) = f_1 f_2(x)$

2. Find the value of k, such that $fog = gof$.

(i) $f(x) = 3x + 2$; $g(x) = 6x - k$

$(fog)\,x = f(g(x))$

$= f[6x - k]$

$= 3(6x - k) + 2$

$= 18x - 3k + 2$

$(gof)\,x = g[f(x)]$

$= g[3x + 2]$

$= 6(3x + 2) - k$

$= 18x + 12 - k$

Given $fog = gof$

$18x - 3k + 2 = 18x + 12 - k$

$-3k + k = 12 - 2$

$-2k = 10$

$k = \frac{10}{-2}$

$\boxed{k = -5}$

(ii) $f(x) = 2x - k$; $g(x) = 4x + 5$

$(fog)\,x = f[g(x)]$

$= f[4x + 5]$

$= 2(4x + 5) - k$

$= 8x + 10 - k$

$(gof)\,x = g[f(x)]$

$= g[2x - k]$

$= 4(2x - k) + 5$

$= 8x - 4k + 5$

Given $fog = gof$

$8 + 10 - k = 8x - 4k + 5$

$-k + 4k = 5 - 10$

$3k = -5$

$\boxed{k = \frac{-5}{3}}$

4. (i) If $f(x) = x^2 - 1$, $g(x) = x - 2$ find a, if $gof(a) = 1$

$f(x) = x^2 - 1$

$f(a) = a^2 - 1$

Given $gof(a) = 1$

$gof(a) \Rightarrow [gof(a)]\,x = 1$

$\Rightarrow g[f(a)\,x] = 1$

$\Rightarrow g[a^2 - 1] = 1$

$a^2 - 1 - 2 = 1$

$a^2 - 3 = 1$

$a^2 = 1 + 3$

$a^2 = 4$

$\boxed{a = \pm 2}$

4. (ii) Find k, if $f(k) = 2k - 1$ and $fof(k) = 5$

Given $fof(k) = 5$

$f[f(k)\,x] = 5$

$f(2k - 1) = 5$

$2(2k - 1) - 1 = 5$

$4k - 2 - 1 = 5$

$4k - 3 = 5$

$4k = 5 + 3$

$4k = 8$

$k = \frac{8}{4}$

$\boxed{k = 2}$

Example 1.22

If $f(x) = 3x - 2$, $g(x) = 2x + k$ and if $fog = gof$, then find the value of k.

$f(x) = 3x - 2$; $g(x) = 2x + k$

$(fog)(x) = f(g(x))$

$= f(2x + k)$

$= 3(2x + k) - 2$

$= 6x + 3x - 2$

$(gof)(x) = g[f(x)]$

$= g[3x - 2]$

$= 2(3x - 2) + k$

$= 6x - 4 + k$

Given $fog = gof$

$6x + 3k - 2 = 6x - 4 + k$

$3k - k = -4 + 2$

$2k = -2$

$$k = \frac{-2}{2}$$

$$\boxed{k = -1}$$

Example 1.23

Find k if fof (k) = 5 where f (k) = 2k − 1.

$$fof(k) = f(f(k))$$
$$= f(2k-1)$$
$$= 2(2k-1) - 1$$
$$= 4k - 2 - 1$$
$$= 4k - 3$$

Given $fof(k) = 5$

$$4k - 3 = 5$$
$$4k = 5 + 3$$
$$k = \frac{8}{4}$$
$$\boxed{k = 2}$$

7. If $f: R \to R$ and $g: R \to R$ are defined by $f(x) = x^5$ and $g(x) = x^4$ then check if f, g are one-one and fog is one-one.

- $f(x) = x^5$

$f(x)$ is **one-one** function. Since every $x \in R$ have unique image.

- Let $f(1) = 1^5 = 1$

$f(-1) = (-1)^5 = -1$

- $g(1) = x^4$

$g(x)$ is **not one-one** function

Let $g(2) = 2^4 = 16$

$g(-2) = (-2)^4 = 16$

$\therefore$ 2, − 2 have same image so it is not one-one.

Now

$$fog = f[g(x)]$$
$$= f(x^4)$$
$$fog = f[g(x)]$$
$$= f(x^4)$$
$$= (x^4)^5$$
$$= x^{20}$$

$(fog)\ x$ is **not one-one** function.

$(fog)(1) = (1)^{20} = 1$

$(fog)(-1) = (-1)^{20} = 1$

$\therefore$ 1, − 1 have same image.

So it is not one-one function.

8. Consider the functions $f(x), g(x), h(x)$ as given below. Show that $(fog)\,oh = fo\,(goh)$ in each case.

(i) $f(x) = x - 1, g(x) = 3x + 1$ and $h(x) = x^2$

LHS

$(fog)\,oh$

$$(fog)\,x = f[g(x)]$$
$$= f(3x+1)$$
$$= 3x + 1 - 1$$
$$= 3x$$
$$(fog)\,oh(x) = (fog)(h(x))$$
$$= (fog)(3x^2)$$
$$= 3x^2 \quad \text{...(1)}$$

RHS

$fo\,(goh)$

$$(goh)\,x = g(h(x))$$
$$= g(x^2) = 3x^2 + 1$$

$$fo\,(goh)\,(x) = f\,(3x^2+1)$$
$$= 3x^2+1-1$$
$$= 3x^2 \quad ...(2)$$

LHS = RHS, Hence $(fog)\,oh = fo\,(goh)$ is verified.

(ii) $f(x) = x^2, g(x) = 2x$ and $h(x) = x+4$

LHS

$(fog)\,oh$

$$(fog)\,x = f\,[g\,(x)]$$
$$= f\,(2x)$$
$$= (2x)^2$$
$$= 4x^2$$
$$(fog)\,oh\,(x) = (fog)\,(x+4)$$
$$= 4\,(x+4)^2$$
$$= 4\,(x^2+8x+16)$$
$$= 4x^2+32x+64 \quad ...(1)$$

RHS

$$(goh)\,(x) = g\,[h\,(x)]$$
$$= g\,(x+4)$$
$$= 2\,(x+4)$$
$$= 2x+8$$
$$fo\,(goh)\,(x) = fo\,[(goh)\,x]$$
$$= f\,(2x+8)$$
$$= (2x+8)^2$$
$$= 4x^2+32x+64 \quad ...(2)$$

LHS = RHS

Hence $(fog)\,oh = fo\,(goh)$ is verified.

(iii) $f(x) = x-4, g(x) = x^2$ and $h(x) = 3x-5$

LHS

$(fog)\,oh$

$$(fog)\,(x) = f\,[g\,(x)]$$
$$= f\,(x^2)$$
$$= x^2-4$$
$$(fog)\,oh\,(x) = (fog)\,(3x-5)$$
$$= (3x-5)^2-4$$
$$= 9x^2-30x+25-4$$
$$= 9x^2-30x+21 \quad ...(1)$$

RHS

$fo\,(goh)$

$$(goh)\,(x) = g\,[h\,(x)]$$
$$= g\,[3x-5]$$
$$= (3x-5)^2$$
$$[fo\,(goh)]\,(x) = fo\,[(goh)\,(x)]$$
$$= f\,(3x-5)^2$$
$$= (3x-5)^2-4$$
$$= 9x^2-30x+25-4$$
$$= 9x^2-30x+21 \quad ...(2)$$

LHS = RHS

Hence $(fog)\,oh = fo\,(goh)$ is verified.

Example 1.24

If $f(x) = 2x+3, g(x) = 1-2x$ and $h(x) = 3x$, prove that $fo\,(goh) = (fog)\,oh$

LHS

$fo\,(goh)$

$$(goh)\,(x) = g\,[h\,(x)]$$
$$= g\,(3x)$$
$$= 1-2\,(3x)$$
$$= 1-6x$$

$[fo\,(goh)]\,(x) = fo\,[(goh)\,(x)]$

$= f\,[1 - 6x]$

$= 2\,(1 - 6x) + 3$

$= 2 - 12x + 3$

$= 5 - 12x \quad ...(1)$

RHS

$[fog]\,oh$

$(fog)\,(x) = f\,[g\,(x)]$

$= f\,(1 - 2x)$

$= (1 - 2x) + 3$

$= 2 - 4x + 3$

$= 5 - 4x$

$[(fog)\,oh]\,(x) = (fog)\,(h\,(x)]$

$= (fog)\,(3x)$

$= 5 - 4\,(3x)$

$= 5 - 12x \quad ...(2)$

From (1) & (2) LHS = RHS

Hence $fo\,(goh) = (fog)\,oh$ is verified.

Example 1.25

Find x if $gff\,(x) = fgg\,(x)$, given $f\,(x) = 3x + 1$ and $g\,(x) = x + 3$

$gff\,(x) = g\,[f\{\,f\,(x)\,\}]$

$= g\,[f\,(3x + 1)]$

$= g\,[3\,(3x + 1) + 1]$

$= g\,(9x + 3 + 1)$

$= g\,(9x + 4)$

$= 9x + 4 + 3$

$= 9x + 7$

$fgg\,(x) = f\,[g\,\{\,g\,(x)\,\}]$

$= f\,[g\,(x + 3)]$

$= f\,[x + 3 + 3]$

$= f\,(x + 6)$

$= 3\,(x + 6) + 1$

$= 3x + 18 + 1$

$= 3x + 19$

Given $gff\,(x) = fgg\,(x)$

$9x + 7 = 3x + 19$

$9x - 3x = 19 - 7$

$6x = 12$

$x = \dfrac{12}{6}$

$\boxed{x = 2}$

Type: II [Problems based on functions]
(QNo: 9,10)

9. Let $f = \{\,(-1, 3), (0, -1), (2, -9)\,\}$ be a linear function from z into z. Find $f\,(x)$

Given $f = \{\,(-1, 3), (0, -1), (2, -9)\,\}$

$\therefore f\,(-1) = 3; f\,(0) = -1; f\,(2) = -9$

$f\,(x) = ax + b$ is the linear equation.

$f\,(-1) = 3$	$f\,(0) = -1$
$a\,(-1) + b = 3$	$a\,(0) + b = -1$
$-a + b = 3 \quad ...(1)$	$b = -1$

$(1) \Rightarrow -a - 1 = 3$

$-a = 3 + 1$

$-a = 4$

$\boxed{a = -4}$

$\therefore f\,(x) = -4x - 1$

10. In electrical circuit theory, a circuit $C\,(t)$ is called a linear circuit if it satisfies the superposition principle given by $C\,(at_1 + bt_2) = a\,C\,(t_1) + b\,C\,(t_2)$ where a, b are constants, show that the circuit $C\,(t) = 3t$ is linear.

Given

$C\,(t) = 3t$

To prove $C(t) = 3t$ is linear

Let $C(at_1) = 3a(t_1)$

$C(bt_2) = 3b(t_2)$

$C(at_1 + bt_2) = 3[at_1 + bt_2]$

$= 3at_1 + 3bt_2$

$= a(3t_1) + b(3t_2)$

$C(at_1 + bt_2) = a[c(t_1) + b[c(t_2)]$

Hence proved.

Exercise 1.6

Multiple choice questions

1. If $n(A \times B) = 6$ and $A = \{1, 3\}$ then $n(B)$ is

1. 1 2. 2 3. 3 4. 6

$n(A \times B) = 6, A = \{1, 3\}$

$\therefore n(A) = 2$

$n(A) \times n(B) = n(A \times B)$

$2 \times n(B) = 6$

$n(B) = \frac{6}{2}$

$n(B) = 3$

Ans. (3) 3

2. $A = \{a, b, p\}, B = \{2, 3\}, C = \{p, q, r, s\}$ then $n[(A \cup C) \times B]$ is

1. 8 2. 20 3. 12 4. 16

Solution:

$A = \{a, b, q\}, B = \{2, 3\}, C = \{p, q, r, s\}$

$A \cup C = \{a, b, p, q, r, s\}$

$\therefore n[(A \cup C) \times B] = n(A \cup C) \times n(B)$

$= 6 \times 2$

$= 12$

Ans. (3) 12

3. If $A = \{1, 2\}, B = \{1, 2, 3, 4\}, C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$ then state which of the following statement is true.

1. $(A \times C) \subset (B \times D)$ 2. $(B \times D) \subset (A \times C)$
3. $(A \times B) \subset (A \times D)$ 4. $(D \times A) \subset (B \times A)$

Solution:

$A = \{1, 2\}, B = \{1, 2, 3, 4\}$

$C = \{5, 6\}, D = \{5, 6, 7, 8\}$

Here option (1) is true

$(A \times C) \subset (B \times D)$

Since All elements in $A \times C$ is in $B \times D$

$A \times C = \{1, 2\} \times \{5, 6\}$

$= \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

$B \times D = \{1, 2, 3, 4\} \times \{5, 6, 7, 8\}$

$= \{(1, 5), (1, 6), (1, 7), (1, 8)$
$(2,5), (2,6), (2,7), (2,8)$
$(3,5), (3,6), (3,7), (3,8)$
$(4, 5), (4, 6), (4, 7), (4, 8)\}$

Here $(A \times C) \in (B \times D)$

Ans. (1) $(A \times C) \subset (B \times D)$

4. If there are 1024 relations from a set $A = \{1, 2, 3, 4, 5\}$ to a set B, then the number of elements in B is

1. 3 2. 2 3. 4 4. 8

Solution:

$A = \{1, 2, 3, 4, 5\}$

Let $n(A) = 5 \rightarrow P$

$n(B) = q$

We have

Total number of relations $= 2^{pq}$

$1024 = 2^{5q}$

$2^{10} = 2^{5q}$

$5q = 10$

$$q = \frac{10}{5}$$

$$\boxed{q = 2}$$

Ans. (2) 2

5. The range of the relations $R = \{ (x, x^2) \mid x$ is a prime number less than 13 $\}$ is

1. $\{ 2, 3, 5, 7 \}$
2. $\{ 2, 3, 5, 7, 11 \}$
3. $\{ 4, 9, 25, 49, 121 \}$
4. $\{ 1, 4, 9, 25, 49, 121 \}$

✍ **Solution:**

$R = \{ (x, x^2) \mid x$ is a prime number less than 13 $\}$

$x = 2, 3, 5, 7, 11$

$x^2 = 4, 9, 25, 49, 121$

$\therefore$ Range $= \{ 4, 9, 25, 4, 9, 121 \}$

Ans. (3) $\{ 4, 9, 25, 49, 121 \}$

6. If the ordered pairs $(a + 2, 4)$ and $(5, 2a + b)$ are equal then (a, b) is

1. $(2, -2)$
2. $(5,1)$
3. $(2,3)$
4. $(3, -2)$

✍ **Solution:**

$(a + 2, 4) = (5, 2a + b)$

$a + 2 = 5$

$a = 5 - 2$

$$\boxed{a = 3}$$

$4 = 2a + b$

$4 = 2\,(3) + b$

$4 = 6 + b$

$4 - 6 = + b$

$-2 = + b$

$$\boxed{b = -2}$$

Ans. (4) $(3, -2)$

7. Let $n(A) = m$ and $n(B) = n$ then the total number of non-empty relations that can be defined from A to B is

1. m^n
2. n^m
3. $2^{mn} - 1$
4. 2^{mn}

✍ **Solution:**

$n(A) = m, n(B) = n$

$\therefore$ Total number of relations $= 2^{mn}$

Ans. (4) 2^{mn}

8. If $\{ (a, 8), (6, b) \}$ represents an identify function, then the value of a and b are respectively

1. (8,6)
2. (8,8)
3. (6,8)
4. (6,6)

✍ **Solution:**

$\{ (a, 8), (6, b) \}$ represents an identity function

$\therefore a = 8$

$b = 6$

Ans. (1) (8, 6)

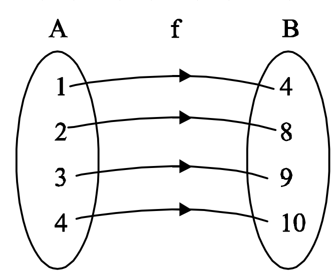
9. Let $A = \{ 1,2, 3, 4 \}$ and $B = \{ 4, 8, 9, 10 \}$. A function $f: A \to B$ given by $f = \{ (1, 4), (2, 8), (3, 9), (4, 10) \}$ is a

1. Many-one function
2. Identify function
3. One-to-one function
4. Into function

✍ **Solution:**

$A = \{ 1, 2, 3, 4 \}; B = \{ 4, 8, 9, 10 \}$

$f = \{ (1, 4), (2, 8), (3, 9), (4, 10) \}$ is



one-to-one and onto function (bijective function)

Ans. (3) one-to-one function

10. If $f(x) = 2x^2$ and $g(x) = \frac{1}{3x}$, then fog is

1. $\frac{3}{2x^2}$ 2. $\frac{2}{3x^2}$ 3. $\frac{2}{9x^2}$ 4. $\frac{1}{6x^2}$

✍ **Solution:**

$f(x) = 2x^2$

$g(x) = \frac{1}{3x}$

$fog = f[g(x)]$

$= f\left(\frac{1}{3x}\right)$

$= 2\left(\frac{1}{3x}\right)^2$

$= \frac{2}{9x^2}$

Ans. (3) $\frac{2}{9x^2}$

---

11. If $f: A \to B$ is a bijective function and if $n(B) = 7$, then $n(A)$ is equal to

1. 7 2. 49 3. 1 4. 14

✍ **Solution:**

$f: A \to B$ is a bijective function

$\therefore n(A) = n(B)$

Here $n(B) = 7$

So $n(A) = 7$ **Ans. (1) 7**

---

12. Let f and g be two functions given by $f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 7)\}$ $g = \{(0, 2), (2, 4), (-4, 2), (7, 0)\}$ then the range of fog is

1. $\{0, 2, 3, 4, 5\}$ 2. $\{-4, 1, 0, 2, 7\}$
3. $\{1, 2, 3, 4, 5\}$ 4. $\{0, 1, 2\}$

✍ **Solution:**

$f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 7)\}$

$g = \{(0, 2), (1, 0), (2, 4), (-4, 2), (7, 0)\}$

Range of $fog = \{0, 1, 2\}$ **Ans. (4) $\{0, 1, 2\}$**

---

13. Let $f(x) = \sqrt{1 + x^2}$ then

1. $f(xy) = f(x) \cdot f(y)$ 2. $f(xy) \geq f(x) \cdot f(y)$
3. $f(xy) \leq f(x) \cdot f(y)$ 4. None of these

✍ **Solution:**

$f(x) = \sqrt{1 + x^2}$, then

$f(xy) \leq f(x) \cdot f(y)$ **Ans. (3) $f(xy) \leq f(x) f(y)$**

---

14. If $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ is a function given by $g(x) = \alpha x + \beta$ then the values of α and β are

1. $(-1, 2)$ 2. $(2, -1)$
3. $(-1, -2)$ 4. $(1,2)$

✍ **Solution:**

$g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$

$g(1) = 1;\ g(2) = 3;\ g(3) = 5;\ g(4) = 7$

Given $g(x) = \alpha x + \beta$

$g(1) = 1$	$g(2) = 3$
$\alpha(1) + \beta = 1$	$\alpha(2) + \beta = 3$
$\alpha + \beta = 1$	$2\alpha + \beta = 3$

$2\alpha + \beta = 3$

$\alpha + \beta = 1$ sub

$\alpha = 2$

Put $\alpha = 2$ in $2 + \beta = 1$

$\beta = 1 - 2$

$\boxed{\beta = -1}$ **Ans. (2) $(2, -1)$**

---

15. $f(x) = (x + 1)^3 - (x - 1)^3$ represents a function which is

1. linear 2. cubic
3. reciprocal 4. quadratic

✍ **Solution:**

$f(x) = (x + 1)^3 - (x - 1)^3$

$= (x^3 + 3x^2 + 3x + 1) - (x^2 - 3x^2 + 3x - 1)$

$= x^3 + 3x^2 + 3x + 1 - x^3 + 3x^2 - 3x + 1$

$= 6x^2 + 2$

Quadratic function **Ans. (4) Quadratic**

UNIT Exercise - I

1. If the ordered pairs $(x^2 - 3x, y^2 + 4y)$ and $(-2, 5)$ are equal, then find x and y.

$$(x^2 - 3x; y^2 + 4y) = (-2, 5)$$

$x^2 - 3x = -2$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x - 1 = 0 \quad x - 2 = 0$

$x = 1 \quad x = 2$

$y^2 + 4y = 5$

$y^2 + 4y - 5 = 0$

$(y+5)(y-1) = 0$

$y + 5 = 0 \quad y - 1 = 0$

$y = -5 \quad y = 1$

2. The cartesian product $A \times A$ has 9 elements among which $(-1, 0)$ and $(0, 1)$ are found. Find the set A and the remaining elements of $A \times A$

$A \times A = 9$ elements

$\therefore n(A) = 3$

Given $(-1, 0)$ and $(0, 1)$

We get $A = \{-1, 0, 1\}$

$\therefore A \times A = \{-1, 0, 1\} \times \{-1, 0, 1\}$

$= \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (1 - 1), (1, 0), (1, 1)\}$

$\therefore$ Remaining elements of $A \times A$

$= \{(-1, -1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0), (1, 1)\}$

3. Given that $f(x) = \begin{cases} \sqrt{x-1} & x \geq 1 \\ 4 & x < 1 \end{cases}$ Find (i) $f(o)$ (ii) $f(3)$ (iii) $f(a+1)$ in terms of a (Given that $a \geq 0$)

$f(x) = \sqrt{x-1}; x = 1, 2, 3 \ldots$

$f(x) = 4; x = 0, -1, -2 \ldots$

(i) $f(0) = 4$

(ii) $f(x) = \sqrt{3-1}$

$= \sqrt{2}$

(iii) $f(a+1) = \sqrt{a+1-1}$

$= \sqrt{a}$

4. Let $A = \{9, 10, 11, 12, 13, 14, 15, 16, 17\}$ and let $f: A \to N$ be defined by $f(n) =$ the highest prime factor of $n \in A$. Write f as a set of ordered pairs and find the range of f.

$A = \{9, 10, 11, 12, 13, 14, 15, 16, 17\}$

$f: A \to N$ defined by

$f(n) =$ height prime factor of $n \in A$

$\therefore f = \{(9, 3), (10, 5), (11, 11), (12, 3), (13, 13), (14, 7), (15, 5), (16, 2), (17, 17)\}$

Range $= \{2, 3, 5, 11, 13, 17\}$

5. Find the domain of the function $f(x) = \sqrt{1 + \sqrt{1 - \sqrt{1 - x^2}}}$

Let

$1 - x^2 = 0$

$x^2 = 1$

$x = \pm 1$

$\therefore$ Domain $= \{-1, 0, 1\}$

Since $\sqrt{1 - x^2} \geq 0$ (positive)

6. If $f(x) = x^2, g(x) = 3x$ and $h(x) = x - 2$, prove that $(fog)\, oh = fo\,(goh)$

$f(x) = x^2; g(x) = 3x; h(x) = x - 2$

$(fog)\, x = f(g(x))$

$= f(3x)$

$= (3x)^2$

$= 9x^2$

$$[(fog)\,oh]\,(x) = (fog)\,(h\,(x))$$
$$= fog\,(x-2)$$
$$= 9\,(x-2)^2$$
$$= 9\,(x^2 - 4x + 4)$$
$$= 9x^2 - 36x + 36 \quad \text{...(1)}$$
$$(goh)\,x = g\,[h\,(x)]$$
$$= g\,[x-2]$$
$$= 3\,(x-2)$$
$$= 3x - 6$$
$$[fo\,(goh)]\,(x) = fo\,[(goh)\,(x)]$$
$$= f\,(3x-6)$$
$$= (3x-6)^2$$
$$= 9x^2 - 36x + 36 \quad \text{...(2)}$$

From (1) & (2)

$(fog)\,oh = fo\,(goh)$ is verified.

---

7. Let $A = \{1, 2\}$ and $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify whether $A \times C$ is a subset of $B \times D$?

$$A \times C = \{1, 2\} \times \{5, 6\}$$
$$= \{(1, 5), (1, 6), (2, 5), (2, 6)\}$$
$$B \times D = \{1, 2, 3, 4\} \times \{5, 6, 7, 8\}$$
$$= \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$$

$\therefore (A \times C) \in (B \times D)$ is verified.

---

8. If $f(x) = \dfrac{x-1}{x+1}, x \neq -1$ show that $f(f(x)) = -\dfrac{1}{x}$, provided $x \neq 0$.

$$f\,[f\,(x)] = f\left[\frac{x-1}{x+1}\right]$$
$$= \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1}$$
$$= \frac{\frac{x-1-(x+1)}{x+1}}{\frac{x-1+(x+1)}{x+1}}$$
$$= \frac{x-1-x-1}{x-1+x+1}$$
$$= \frac{-2}{2x}$$
$$f\,[f\,(x)] = \frac{-1}{x} \text{ proved}$$

---

9. The functions f and g are defined by $f(x) = 6x + 8;\ g(x) = \dfrac{x-2}{3}$

(i) Calculate the value of $gg\left(\dfrac{1}{2}\right)$

(ii) Write an expression for $gf(x)$ in its simplest form.

$$f\,(x) = 6x + 8;\ g\,(x) = \frac{x-2}{3}$$

(i) $gg\,(x) = g\,[g\,(x)]$

$$= g\left[\frac{x-2}{3}\right]$$
$$= \frac{\frac{x-2}{3} - 2}{3}$$
$$= \frac{x-2-6}{3 \times 3}$$
$$= \frac{x-8}{9}$$
$$gg\left(\frac{1}{2}\right) = \frac{\frac{1}{2} - 8}{9}$$
$$= \frac{-15}{2 \times 9}$$
$$= \frac{-5}{6}$$

(ii) $gf(x) = g[f(x)]$

$= g[6x+8]$

$= \dfrac{6x+8-2}{3}$

$= \dfrac{6x+6}{3}$

$= \dfrac{6(x+1)}{3}$

$= 2(x+1)$

---

10. Write the domain of the following real functions.

(i) $f(x) = \dfrac{2x+1}{x-9}$

Let $x - 9 = 0$

$x = 9$

$\therefore$ Domain $= R - \{9\}$

(ii) $P(x) = \dfrac{-5}{4x^2+1}$

Here

$4x^2 + 1 > 0$ for all $x \in R$

$\therefore$ Domain $= R$

(iii) $g(x) = \sqrt{x-2}$

Here $\sqrt{x-2} \geq 0$ [should be positive]

$\therefore$ Domain $= [2, \infty]$

(iv) $h(x) = x + 6$

Domain $= R$

CHAPTER 2

NUMBERS AND SEQUENCES

Exercise 2.1

KEY POINTS

Euclid's Division Lemma

Let a and $b\,(a > b)$ be any two positive integers. Then, there exist unique integers q and r such that

$$\boxed{a = bq + r, 0 \le r < b}$$

Note:

- The remainder is always less than the division.
- If $r = 0$ then $a = bq$. So 'b' divides 'a'.
- Similarly if b divides a then $a = bq$
- Generalised form of Euclid's division lemma $a = bq + r, 0 \le r < |b|$
- Euclid's division algorithm provides an easier way to compute the thighest common factor (HCF) of two given positive integers.
- Two positive integers are said to be **relatively prime** or **Co-prime** if then thighest common factor is 1.

Type I: Problems based on Euclid's division lemma.

Example 2.1, 2.2, Q.No. 1, 2, 3, 4, 5, Example 2.3

Example 2.1

We have 34 cakes, Each box can hold 5 cakes only. How many boxes we need to pack and how many cakes are unpacked

$$34 = 5 \times 6 + 4$$

$$a = bq + r$$

$a -$ dividend

$b -$ divisior

$r -$ remainder

$q -$ quotient

$\therefore$ 6 boxes needed and 4 cakes are unpacked.

Example 2.2

Find the quotient and remainder when a is divided by b in the following cases (i) $a = -12, b = 5$ (ii) $a = 17, b = -3$ (iii) $a = -19, b = -4$

(i) $a = -12, b = 5$

By Euclid's division lemme

$a = bq + r$, where $0 \le r < |b|$

$-12 = 5 \times (-3) + 3, 0 \le r < |5|$

$\therefore$ Quotient $q = -3$, Remainder $r = 3$

(ii) $a = 17, b = -3$

By Euclid's division lemma

$a = bq + r$, where $0 \le r < |b|$

$17 = (-3) \times (-5) + 2, 0 \le r \le |-3|$

$\therefore$ Quotient $q = -5$

Remainder $r = 2$

(iii) $a = -19, b = -4$

By Euclid's division lemma

$a = bq + r$, where $0 \le r < |b|$

$-19 = (-4)(5) + 1, 0 \le r < |-4|$

$\therefore$ Quotient $q = 5$

Remainder $r = 1$

1. Find all positive integers, when divided by 3 leaves remainder 2.

Given $b = 3$; $r = 2$

by Euclid's division lemma

$$a = bq + r, 0 \le r < |\, b \,|$$

$$a = 3q + 2, 0 \le q < |\, 3 \,|$$

$\therefore$ All positive lutegers, when divided by 3 leaves remainder 2 is 2, 5, 8, 11 ...

---

2. A man has 532 flower pots. He wants to arrange them in rows such that each row contains 21 flower pots. Find the number of completed rows and how many flower pots are left over.

Here $a = 532$

$b = 21$

by Euclid's division lemma

$$a = bq + r, 0 \le r \le |\, b \,|$$

$$522 = 21 \times 25 + 7$$

$\therefore$ No. of completed rows = 25

Left over thower pots = 7 pots

---

3. Prove that product of two consecutive positive integers is divisible by 2.

Let two consecutive positive integer $n, n+1$.

Product $= n \times (n+1)$

$= n^2 + n$

Positive integers divisible by 2 is of the form $2q$ or $2q + 1$

Case (i) When $n = 2q$

$$n^2 + n = (2q)^2 + 2q$$

$$= 4q^2 + 2q$$

$$= 2q\,(2q+1)$$

$$= 2m \quad \text{where } m = q\,(2q+1)$$

which is divisible by 2.

Case (ii) When $n = 2q + 1$

$$n^2 + n = (2q+1)^2 + (2q+1)$$

$$= 4q^2 + 4q + 1 + 2q + 1$$

$$= 4q^2 + 6q + 2$$

$$= 2\,(2q^2 + 3q + 1)$$

$$= 2\text{ m}$$

$$= \text{where } m = 2q^2 + 3q + 1$$

which is divisible by 2.

Hence proved.

---

4. When the positive integers be a, b and c divided by 13, the repective remainders are 9, 7 and 10. Show that $a + b + c$ is divisible by 13.

Let the positive integers a, b, c

Given

$$a = 13q + 9$$

$$b = 13q + 7$$

$$c = 13q + 10$$

$$\therefore \quad a + b + c = 13q + 9 + 13q + 7 + 13q + 10$$

$$= 39q + 26$$

$$= 13\,(3q + 2)$$

$$= 13m \quad (\text{where } m = 3q + 2)$$

Which is divisible by 13.

Hence proved.

---

5. Prove that square of any integer leaves the remainder either 0 or 1 when divided by 4.

Let x be any integer its square is x^2

Case (i) When $x = \text{even}$

Let $x = 2m$

$$x^2 = 4m^2$$

$$x^2 = 4m^2 + 0$$

Which is divisible by 4 leaves remainder 0.

Case (ii): When $x = \text{odd}$

Let $x = 2m + 1$

$$x^2 = (2m+1)^2$$
$$= 4m^2 + 4m + 1$$
$$= 4m(m+1) + 1$$

$x^2 = 4q + 1$ when $q = m(m+1)$

Which is divisible by 4 leaves remainder 1.

Type II: Find H.C.F using Euclid's Division Algorithm - Problems

Q.No. 6 (i) (ii) (iii) (iv), Example 2.6, 7, Example 2.4, 2.5, 8, 9, 10

6. Use Euclid's Division Algorithm to find the Highest Common Factor (HCF) of

(i) 340 and 412

To find the H.C.F of 340 and 412 using Euclid's division algorithm.

- $412 = 340 \times 1 + 72$

  The remainder $72 \neq 0$
- Again Applying Euclid's division algorithm

  $$340 = 72 \times 4 + 52$$

  The remainder $52 \neq 0$
- Again Applying Euclid's division algorithm

  $$72 = 52 \times 1 + 20$$

  The remainder $20 \neq 0$
- Again Applying Euclid's division algorithm

  $$52 = 20 \times 2 + 12$$

  The remainder $12 \neq 0$
- Again Applying Euclid's division algorithm

  $$20 = 12 \times 1 + 8$$

  The remainder $8 \neq 0$
- Again Applying Euclid's division algorithm

  $$12 = 8 \times 1 + 4$$

  The remainder $4 \neq 0$
- Again Applying Euclid's division algorithm

  $$8 = 4 \times 2 + 0$$

  The remainder is 0

  $\therefore$ HCF of 340 and 412 is 4

(ii) 867 and 255

Use Euclid's division algorithm till we get remainder is zero.

- $867 = 255 \times 3 + 102$

  $R = 102 \neq 0$
- $255 = 102 \times 2 + 51$

  $R = 51 \neq 0$
- $102 = 51 \times 2 + 0$

  $R = 0$

  $\therefore$ HCF of 867 and 255 is 51.

(iii) 10224 and 9648

Use Euclid's division algorithm till we get remainder is zero.

- $10224 = 9648 \times 1 + 576$

  $R = 576 \neq 0$
- $9648 = 576 \times 16 + 432$

  $R = 432 \neq 0$
- $576 = 432 \times 1 + 144$

  $R = 144 \neq 0$
- $432 = 144 \times 3 + 0$

  $R = 0$

  $\therefore$ HCF of 10224 and 9648 is 144.

(iv) 84, 90 and 120

Let two numbers 84, 90 use Euclid's division algorithm till we get remainder is zero.

- $90 = 84 \times 1 + 6$

  $R = 6 \neq 0$
- $84 = 6 \times 14 + 0$

  $R = 0$

$\therefore$ HCF of 90, 84 is 6.

Now let 6 and 120 use Euclid's division algorithm till we get remainder is zero.

$$120 = 6 \times 20 + 0$$

$$R = 0$$

$\therefore$ HCF of 120 and 6 is 6.

Hence HCF of 84, 90 and 120 is 6.

Example 2.6

Find the HCF of 396, 504, 636.

✍ **Solution:**

Let two numbers 396 and 504 use Euclid's division algorithm till we get remainder is zero.

- $504 = 396 \times 1 + 108$

  $R = 108 \neq 0$
- $396 = 108 \times 3 + 72$

  $R = 72 \neq 0$
- $108 = 72 \times 1 + 36$

  $R = 36 \neq 0$
- $72 = 36 \times 2 + 0$

  $R = 0$

$\therefore$ HCF of 396 and 504 is 36.

Now find HCF of 636 and 36 use Euclid's division algorithm till we get remainder is zero.

- $636 = 36 \times 17 + 24$

  $R = 24 \neq 0$
- $36 = 24 \times 1 + 12$

  $R = 12 \neq 0$
- $24 = 12 \times 2 + 0$

  $R = 0$

$\therefore$ HCF of 636 and 36 is 12.

Hence HCF of 396, 504, 636 is 12.

---

7. Find the largest number which divides 1230 and 1926 leaving remainder 12 in each case.

Since the remainder is 12, the required number is the HCF of the number

$$1230 - 12 = 1218$$

$$1926 - 12 = 1914$$

Use Euclid's division algorithm till we get remainder is zero.

- $1914 = 1218 \times 1 + 696$

  $R = 696 \neq 0$
- $1218 = 696 \times 1 + 522$

  $R = 522 \neq 0$
- $696 = 522 \times 1 + 174$

  $R = 174 \neq 0$
- $522 = 174 \times 3 + 0$

  $R = 0$

$\therefore$ HCF of 1218 and 1914 is 174.

Hence required largest number is 174.

Example 2.4

If the Highest Common Factor of 210 and 55 is expressible in the form $55x - 325$, find x.

Using Euclid's Division Algorithm let us find the HCF of 210 and 55.

- $210 = 55 \times 3 + 45$

  $R = 45 \neq 0$
- $55 = 45 \times 1 + 10$

  $R = 10 \neq 0$
- $45 = 10 \times 4 + 5$

  $R = 5 \neq 0$
- $10 = 5 \times 2 + 0$

  $R = 0$

$\therefore$ HCF of 210 and 55 is 5.

Given $55x - 325 = \text{HCF}$

$$55x - 325 = 5$$
$$55x = 5 + 325$$
$$55x = 330$$
$$x = \frac{330}{55}$$
$$\boxed{x = 6}$$

Example 2.5

Find the greatest number that will divide 445 and 572 leaving remainders 4 and 5 respectively.

Since the remainders are 4, 5 respectively the required number is the HCF of the number.

$445 - 4 = 441,\ 572 - 5 = 567.$

To find HCF of 441 and 567 use Euclid's Division Algorithm till we get remainder zero.

- $567 = 441 \times 1 + 126$

  $R = 126 \neq 0$
- $441 = 126 \times 3 + 63$

  $R = 63 \neq 0$
- $126 = 63 \times 2 + 0$

  $R = 0$

$\therefore$ HCF of 441, 567 is 63 and so the required number is 63.

8. If d is the Highest Common Factor of 32 and 60, find x and y satisfying $d = 32x + 60y$.

To find HCF of 32 and 60 use Euclid's division Algorithm till we get remainder is zero.

- $60 = 32 \times 1 + 28$

  $R = 28 \neq 0$ ...(i)
- $32 = 28 \times 1 + 4$

  $R = 4 \neq 0$ ...(ii)
- $28 = 4 \times 7 + 0$

  $R = 0$ ...(iii)

$\therefore$ HCF of 32 and 60 is 4.

From (ii), we get

$$32 = 28 \times 1 + 4$$
$$4 = 32 - 28 \times 1$$
$$4 = 32 - 28$$
$$4 = 32 - (60 - 32)$$
$$4 = 32 - 60 + 32$$
$$4 = 2\,(32) - 60$$
$$4 = (32 \times 2) + (-1) \times 60$$

Given

$$d = 32x + 60y$$
$$\therefore\ x = 2$$
$$y = -1$$

9. A positive integer when divided by 88 gives the remainder 61. What will be the remainder when the same number is divisible by 11?

Let the number x (positive)

Given

$x = 88y + 61$

x is divisible by 11, also 88 is divisible by 11

$$\therefore\ 61 = 11 \times 5 + 6$$

Here remainder is 6.

Hence required remainder is 6.

10. Prove that two consective positive integers are always co-prime.

Let two consecutive numbers

$$n, n + 1$$

use Euclid's division algorithm to find HCF of $n, n + 1$

- $n + 1 = n \times 1 + 1$

  Remainder is $1 \neq 0$
- $n = n \times 1 + 0$

Remainder is 0

$\therefore$ HCF of $n, n+1$ is 1.

Hence

$\therefore$ The two consecutive numbers are always co prime.

Exercise 2.2

KEY POINTS

Fundamental Theorem of Arithmetic

Every natural number except 1 can be factorized as a product of primes and this factorization is unique except for the order in which the prime factors are written.

Note:

- If a prime number P divides ab then either P divides a or P divides b. That is P divides at least one of them.
- If a composite number n divides ab, then n neither divide a nor b

Example

6 divides 4×3 but 6 neither divide 4 nor 3.

Type I: Find LCM and HCF using fundamental theorem of arithmetic

Q.No. 3, 6, 7, 8, 9

3. Find the HCF of 252525 and 363636

$$\begin{array}{r|r} 5 & 252525 \\ \hline 5 & 50505 \\ \hline 3 & 10101 \\ \hline 7 & 3367 \\ \hline 13 & 481 \\ \hline & 37 \end{array} \qquad \begin{array}{r|r} 2 & 363636 \\ \hline 2 & 181818 \\ \hline 3 & 90909 \\ \hline 3 & 30303 \\ \hline 3 & 10101 \\ \hline 7 & 3367 \\ \hline 13 & 481 \\ \hline & 37 \end{array}$$

$252525 = 5\times 5\times 3\times 7\times 13\times 37$

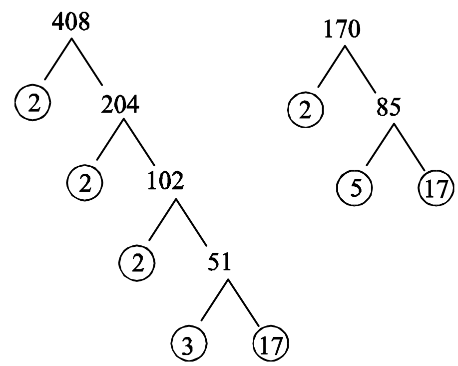
$363636 = 2\times 2\times 3\times 3\times 3\times 7\times 13\times 37$

HCF $= 3\times 7\times 13\times 37$

$= 10{,}101$

---

6. Find the LCM and HCF of 408 and 170 by applying the fundamental theorem of arithmetic.



$408 = 2\times 2\times 2\times 3\times 17$

$170 = 2\times 5\times 17$

HCF $= 2\times 17 = 34$

LCM $= 2\times 17\times 2\times 2\times 3\times 5 = 2040$

---

7. Find the greatest number consisting of 6 digits which is exactly divisible by 24, 15, 36?

First find LCM of 24, 15, 36

$$\begin{array}{r|l} 3 & 24,\ 15,\ 36 \\ \hline 2 & 8,\ 5,\ 12 \\ \hline 2 & 4,\ 5,\ 6 \\ \hline & 2,\ 5,\ 3 \end{array}$$

LCM $= 3\times 2\times 2\times 2\times 5\times 3$

$= 360$

Greatest 6 digit number is 999999

$$\begin{array}{r|r} & 2777 \\ 360 & 999999 \\ & 720 \\ \hline & 2799 \\ & 2520 \\ \hline & 2799 \\ & 2520 \\ \hline & 2799 \\ & 2520 \\ \hline & 279 \end{array}$$

$\therefore\ 999999 - 279 = 999720.$

Hence greatest 6 digit number which is exactly divisible by 24, 15, 36 is 999720.

8. What is the smallest number that when divided by three numbers such as 35, 56 and 91 leaves remainder 7 in each case.

$$\begin{array}{r|l} 7 & 35,\ 56,\ 91 \\ \hline & 5,\ 8,\ 13 \end{array}$$

$$\text{LCM} = 7 \times 5 \times 8 \times 13$$

$$= 3640$$

Required number $= 3640 + 7 = 3647$

Which leaves remainder 7 in each case.

---

9. Find the least number that is divisible by the first ten natural numbers.

Let first 10 natural numbers and its prime factors

$$1 = 1$$
$$2 = 2$$
$$3 = 3$$
$$4 = 2 \times 2$$
$$5 = 5$$
$$6 = 2 \times 3$$
$$7 = 7$$
$$8 = 2 \times 2 \times 2$$
$$9 = 3 \times 3$$
$$10 = 2 \times 5$$

$\text{LCM} = 2^3 \times 3^2 \times 5 \times 7$

$= .2520$

Type II: Find unknown values

Q.No. 1, 2, Example 2.8, 4, 5, Example 2.7, 2.9 and 2.10

1. For what values of natural number n, 4^n can end with the digit 6?

$4^n = (2 \times 2)^n$

$= 2^n \times 2^n$

2 is a factor of 4 and 4^n is always even and end with 4 and 6.

When n is an even number say 2, 4, 6, 8 then 4^n can end with the digit 6.

Example:

$$4^2 = 16$$
$$4^4 = 256$$
$$4^6 = 4096$$
$$4^8 = 65.536$$

---

2. If m, n are natural numbers, for what values of m, deos $2^n \times 5^m$ ends in 5?

- 2^n end with always even for all values of n (2, 4, 6, 8)
- 5^m end with always with 5 for all values of m

But $2^n \times 5^m$ is always even and ends in 0.

$\therefore$ No value of m will satisfy $2^n \times 5^m$ ends in 5.

Example 2.8

Can the number 6^n, n being a natural number end with the digit 5? Give reason for your answer.

$$6^n = (2 \times 3)^n = 2^n \times 3^n$$

2 is a factor of 6^n. So 6^n is always even.

But any number whose end digit 5 is always odd.

$\therefore$ Hence 6^n cannot end with the digit 5.

4. If $13824 = 2^a \times 3^b$ then find 'a' and 'b'

$$\begin{array}{r|r}
2 & 13824 \\ \hline
2 & 6912 \\ \hline
2 & 3456 \\ \hline
2 & 1728 \\ \hline
2 & 864 \\ \hline
2 & 432 \\ \hline
2 & 216 \\ \hline
2 & 108 \\ \hline
2 & 54 \\ \hline
3 & 27 \\ \hline
3 & 9 \\ \hline
 & 3
\end{array}$$

$\therefore \quad 13824 = 2^9 \times 3^3$

Given $13824 = 2^a \times 3^b$

$\therefore \quad \mathbf{a = 9 \quad b = 3}$

Example 2.10

'a' and 'b' are two positive integers such that $a^b \times b^a = 800$. Find 'a' and 'b'

$$\begin{array}{r|r}
2 & 800 \\ \hline
2 & 400 \\ \hline
2 & 200 \\ \hline
2 & 100 \\ \hline
2 & 50 \\ \hline
2 & 25 \\ \hline
 & 5
\end{array}$$

$$800 = 2^5 \times 5^2 \text{ (or) } 5^2 \times 2^5$$

Given $800 = a^b \times b^a$

$\therefore$ We get

$a = 2, \quad b = 5$

(or) $\quad a = 5, \quad b = 2$

5. If $P_1^{x_1} \times P_2^{x_2} \times P_3^{x_3} \times P_4^{x_4} = 113400$ where P_1, P_2, P_3, P_4 are primes in ascending order and x_1, x_2, x_3, x_4 are integers, find the value of P_1, P_2, P_3, P_4 and x_1, x_2, x_3, x_4

$$\begin{array}{r|r}
2 & 113400 \\ \hline
2 & 56700 \\ \hline
2 & 28350 \\ \hline
3 & 14175 \\ \hline
3 & 4725 \\ \hline
3 & 1575 \\ \hline
3 & 525 \\ \hline
5 & 175 \\ \hline
5 & 35 \\ \hline
 & 7
\end{array}$$

Where P_1, P_2, P_3, P_4 are primes in ascending order

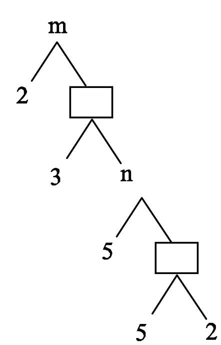
$$113400 = 2^3 \times 3^4 \times 5^2 \times 7^1$$

Comparing with $= P_1^{x_1} \times P_2^{x_2} \times P_3^{x_3} \times P_4^{x_4}$

$$\therefore \begin{array}{l|l}
P_1 = 2 & x_1 = 3 \\
P_2 = 3 & x_2 = 4 \\
P_3 = 5 & x_3 = 2 \\
P_4 = 7 & x_4 = 1
\end{array}$$

Example 2.7

In the given factor tree find the numbers m and n



Value of the first bore from bottom

$$= 5 \times 2 = 10$$

Value of $\quad n = 5 \times 10 = 50$

Value of the second box from bottom

$$= 3 \times 50 = 150$$

Value of $m = 2 \times 150 = 300$

$$\therefore \quad m = 300$$

$$n = 50$$

Example 2.9

Is $7 \times 5 \times 3 \times 2 + 3$ a composite number? Justify your answer.

Yes, the given number is a composite number, because

$7 \times 5 \times 3 \times 2 + 3$

$= 3\ (7 \times 5 \times 2 + 1)$

$= 3 \times 71$

Since the given number can be factorized in terms of two primes it is a **composite number.**

Exercise 2.3

KEY POINTS

1. Modular Arithmetic

In a clock the numbers wrap around 1 to 12. This type of wrapping around after hitting some value is called **"Modular Arithmetic."**

In mathematics, modular arithmetic is a system of arithmetic for integers where numbers wrap around a certain value.

2. Congruence Modular

Two integers a and b are congruence modulo 'n' if they differ by an integer multiple of n. That $b - a = kn$ for some integer k. This can also be written as $a = b$ (mod n). Here the number n is called modulus. In other words,

$a \equiv b$ (mod n) means $a - b$ is divisible by n for example, $61 \equiv 5$ (mod 7) because $61 - 5 = 56$ is divisible by 7.

Note:

(i) When a positive integer is divided by n, then the possible remainders are 0, 1, 2 ... $n - 1$.

(ii) Thus, when we work with modulo n, we replace all the numbers by their remainders upon division by n, given by 0, 1, 2, 3... $n - 1$.

3. Connecting Euclid's Division Lemma and Modular Arithmetic

The equation $n = mq + r$ through Euclid's division lemma can be written as $n \equiv r$ (mod m)

4. Modulo Operations

1. a, b, c and d are integers and m is a positive integer such that if $a \equiv b$ (mod m) and $c = d$ (mod m) then

   (i) $(a + c) \equiv (b + d)$ (mod m)

   (ii) $(a - c) \equiv (b - d)$ (mod m)

   (iii) $a \times c) \equiv (b \times d)$ (mod m)

2. If $a \equiv b$ (mod m) then

   (i) $ac \equiv bc$ (mod m)

   (ii) $a \pm c = b \pm c$ (mod m) for any integer 'c'

Type I: Solve based on modulus sums

Q.No. 1(i)(ii)(iii)(iv)(v), 2, 3, 4, Example 2.12, 2.13(i)(ii), 2.14, 2.15, 2.16, 2.11

1. Find the least positive value of x such that

(i) $71 \equiv x$ (mod 8)

$71 \equiv x$ (mod 8)

$71 - x = 8n$ for some integer n

Put $x = 7$ we get 64 is a multiple of 8

$\therefore \quad 71 \equiv 7$ (mod 8)

Hence $\boldsymbol{x = 7}$

(ii) $78 + x \equiv 3$ (mod 5)

$$78 + x \equiv 3 \text{ (mod 5)}$$

$78 + x - 3 = 5n$ for some integer n

$75 + x = 5n$

$75 + x$ is a multiple of 5.

Put $x = 5$, we get 80 is a multiple of 5

$\therefore$ least value of x is 5.

(iii) $89 \equiv (x + 3) \pmod 4$

$89 \equiv (x + 3) \pmod 4$

$89 - (x + 3) = 4n$ for some integer 'n'

$89 - x - 3 = 4n$

$86 - x = 4n$

$86 - x$ is a multiple of 4

Let $x = 2$, we get $86 - 2 = 84$ is a multiple of 4.

$\therefore$ Least value of x is 2.

(iv) $96 \equiv \frac{x}{7} \pmod 5$

$96 \equiv \frac{x}{7} \pmod 5$

$96 - \frac{x}{7} = 5n$ for some integer 'n'

$\frac{672 - x}{7} = 5n$

$672 - x = 35n$

$672 - x$ is a multiple of 35

Put $x = 7$, we get $672 - 5 = 665$ is a multiple of 35.

$\therefore$ Least value of x is 7.

(v) $5x \equiv 4 \pmod 6$

$5x \equiv 4 \pmod 6$

$5x - 4 = 6n$ for some integer 'n'

Put $x = 2$, we get $5(2) - 4 = 6$ is a multiple of 6

$\therefore$ Least value of x is 2

---

2. If x is congruent to 13 modulo 17 then $7x - 3$ is congruent to which number modulo 17?

Given $x \equiv 13 \pmod{17}$ ...(1)

$7x - 3 = P \pmod{17}$

$7x - 3 \equiv P \pmod{17}$ ...(2)

From (1)

$x \equiv 13 \pmod{17}$

$x - 13 = 17n$ for some integer 'n'.

$x - 13$ is a multiple of 17

Put $x = 30$, $30 - 13 = 17$ is a multiple of 17

$\therefore$ Least value of x is 30.

(1) $\Rightarrow$ $30 = 13 \pmod{17}$

(2) $\Rightarrow$ $7x - 3 \equiv P \pmod{17}$

$7(30) - 3 \equiv P \pmod{17}$

$207 \equiv P \pmod{17}$

$207 - P = 17n$ for some integer 'n'

Put $n = 3$, we get $207 - 3 = 204$ is a multiple of 17.

$$\boxed{P = 3}$$

---

3. Solve $5x \equiv 4 \pmod 6$

$5x \equiv 4 \pmod 6$

$5x - 4 = 6n$ for some integer 'n'

$5x = 6n + 4$

$$x = \frac{6n + 4}{5}$$

Where $n = 1, 6, 11 \ldots$, we get

$x = 2, 8, 14 \ldots$

---

4. Solve $3x - 2 \equiv 0 \pmod 1)$

$3x - 2 \equiv 0 \pmod{11}$

$3x - 2 = 11n$ for some integer 'n'

$3x - 2$ is a multiple of 11.

$$3x = 11n + 2$$

$x = \dfrac{11n+2}{3}$ where $n = 2, 5, 8 \ldots$

$x = \dfrac{11\,(2)+2}{3} = \dfrac{24}{3} = 8$

$x = \dfrac{11\,(5)+2}{3} = \dfrac{57}{3} = 19$

$x = \dfrac{11\,(8)+2}{3} = \dfrac{90}{3} = 30$

$x = 8, 19, 30, \ldots$

Example 2.12

Determine the value of d such that $\mathbf{15 \equiv 3 \ (mod\ \mathit{d})}$

$15 \equiv 3 \pmod d$

$15 - 3 = kd$ for some integer 'k'

$12 = kd$

Gives d divides 12.

The divisors of 12 are 1, 2, 3, 4, 6, 12. But d should be larger than 3 and so possible values for d are 4, 6, 12.

Example 2.13

Find the least positive value of x such that

(i) $\mathbf{67 + x \equiv 1 \ (mod\ 4)}$

$67 + x \equiv 1 \pmod 4$

$67 + x - 1 = 4n$ for some integer 'n'

$66 + x = 4n$

$66 + x$ is a multiple of 4.

Put $x = 2$, we get $66 + 2 = 68$ is a multiple of 4.

$\therefore$ Least value of x is 2

(ii) $\mathbf{98 \equiv (x + 4) \ (mod\ 5)}$

$98 \equiv (x + 4) \pmod 5$

$98 - (x + 4) = 5n$, for some integer 'n'.

$98 - x - 4 = 5n$

$94 - x = 5n$

$94 - x$ is a multiple of 5

Put $x = 4$ we get $94 - 4 = 90$ is a multiple of 5.

$\therefore$ Least value of x is 4.

Example 2.14

Solve $\mathbf{8x \equiv 1 \ (mod\ 11)}$

$8x \equiv 1 \pmod{11}$

$8x - 1 = 11n$ for some integer 'n'

$8x = 11n + 1$

$x = \dfrac{11n+1}{8}$ where $n = 5, 13, 21, 29 \ldots$

$x = \dfrac{11\,(5)+1}{8} = \dfrac{56}{8} = 7$

$x = \dfrac{11\,(13)+1}{8} = \dfrac{144}{8} = 18$

$x = \dfrac{11\,(21)+1}{8} = \dfrac{232}{8} = 29$

$x = \dfrac{11\,(29)+1}{8} = \dfrac{320}{8} = 40$

$\therefore \quad x = 7, 18, 29, 40, \ldots$

Example 2.15

Compute x such that $\mathbf{10^4 \equiv x \ (mod\ 19)}$

$$10^4 \equiv x \pmod{19}$$

We have $10^2 = 100 \equiv 5 \pmod{19}$

$$(10^2)^2 = (100)^2 \equiv 5^2 \pmod{19}$$

$10^4 = 25 \pmod{19}$

$10^4 = 6 \pmod{19}$

$\therefore \quad \mathbf{x = 6}$ Since $25 \equiv 6 \pmod{19}$

Example 2.16

Find the number of integer solution of **$3x = 1$ (mod 15)**

$$3x \equiv 1 \pmod{15}$$

$3x - 1 = 15n$ for some integer

$$3x = 15n + 1$$

$$x = \frac{15n+1}{3}$$

$$x = \frac{15n}{3} + \frac{1}{3}$$

$$x = 5n + \frac{1}{3}$$

Here $5n$ is an integer.

$5n + \frac{1}{3}$ cannot be integer.

$\therefore$ There is no integer solution for n.

Example 2.11

Find the remainders when 70004 and 778 is divisible by 7.

- Since 70000 divisible by 7

$$70000 \equiv 0 \pmod 7$$

$$70000 + 4 = 0 + 4 \pmod 7$$

$$70004 = 4 \pmod 7$$

$\therefore$ The remainder when 70004 is divisible by 7 is 4.

- Since 777 is divisible by 7

$$777 \equiv 0 \pmod 7$$

$$777 + 1 \equiv 0 + 1 \pmod 7$$

$$778 \equiv 1 \pmod 7$$

$\therefore$ The remainder when 778 is divided by 7 is 1.

Type II: [Word Problems]

Q.No. 5, 6, 7, 8, 9, 10 Example 2.17, 2.18

5. What is the time 100 hours after 7 am?

$100 \equiv x \pmod{12}$ [7 comes in every 12 hrs]

$100 - x = 12n$ for some integer 'n'

$100 - x$ is a multiple of 12

Put $x = 4$ we get $100 - 4 = 96$ is a multiple of 12

$\therefore$ Least value of x is 4

$\therefore$ The time 100 hrs after 7 o'clock is $7 + 4 = 11$ o'clock i.e. 11 am.

---

6. What is the time 15 hours before 11 pm?

$15 \equiv x \pmod{12}$

$15 - x = 12n$ for some integer 'n'

$15 - x$ is a multiple of 12

Put $x = 3$ we get $15 - 3 = 12$ is a multiple of 12

$\therefore$ Least value of x is 3.

The time 15 hours before 11 pm is $11 - 3 = 8$ pm.

---

7. Today is Tuesday. My uncle will come after 45 days. In which day my uncle will be coming?

$45 \equiv x \pmod 7$ [No. of days in a week is 7]

$45 - x = 7n$ for some integer 'n'

$45 - x$ is a multiple of 7

Put $x = 3$, we get $45 - 3 = 42$

$\therefore$ Least value of x is 3

Three days after Tuesday is Friday.

$\therefore$ Uncle will come on Friday.

Type II: Find the indicated terms
Q.No. 4 (i) (ii), 5, Example 2.21

4. Find the indicated terms of the sequences whose n^th terms are given by

(i) $a_n = \frac{5n}{n+2}$; a_6 and a_{13}

$$a_n = \frac{5n}{n+2}$$

$$a_6 = \frac{5(6)}{6+2} = \frac{30}{8} = \frac{15}{4}$$

$$a_{13} = \frac{5(13)}{13+2} = \frac{65}{15} = \frac{13}{3}$$

(ii) $a_n = -(n^2 - 4)$; a_4 and a_{11}

$$a_n = -(n^2 - 4)$$

$$a_4 = -(4^2 - 4) = -(16 - 4) = -12$$

$$a_{11} = -(11^2 - 4) = -(121 - 4) = -117 .$$

---

5. Find a_8 and a_{15} whose n^th term is

$$a_n = \begin{cases} \frac{n^2 - 1}{n+3} \text{ ; } n \text{ is even, } n \in N \\ \frac{n^2}{2n+1} \text{ ; } n \text{ is odd ; } n \in N \end{cases}$$

- a_8 is even term, so

Let $a_n = \frac{n^2 - 1}{n+3}$

$$a_8 = \frac{8^2 - 1}{8+3} = \frac{64-1}{11} = \frac{63}{11}$$

- a_{15} is odd term so

Let $a_n = \frac{n^2}{2n+1}$

$$a_{15} = \frac{15^2}{2(15)+1} = \frac{225}{30+1} = \frac{225}{31}$$

Hence,

$$\therefore \quad a_8 = \frac{63}{11} \text{ and } a_{15} = \frac{225}{31}$$

Example 2.21

The general term of a sequence is defined as $a_n = \begin{cases} n(n+3) \text{ ; } n \in N \text{ is odd} \\ n^2 + 1 \text{ ; } n \in N \text{ is even} \end{cases}$ ***Find the eleventh and eighteeth terms.***

- a_{11} is odd term, so

Let $a_n = n(n+3)$

$$a_{11} = 11(11+3) = 11 \times 14 = 154$$

- a_{18} is even term, so

Let $a_n = n^2 + 1$

$$a_{18} = 18^2 + 1 = 324 + 1 = 325$$

Exercise 2.5

KEY POINTS

1. Arithmetic Progression (A.P)

Let a and d be real numbers. Then the numbers of the form $a, a+a, a+2d, a+3d, \ldots$ is said to form Arithmetic Progression denoted by A.P.

(i) First term is denoted by 'a' and common difference is denoted by 'd' where $\boldsymbol{d = t_2 - t_1}$

(ii) To check given sequence is A.P or not $\boldsymbol{t_2 - t_1 = t_3 - t_2}$

(iii) n^th term or general term of an A.P. $\boldsymbol{t_n = a + (n-1)d}$

(iv) General form of an A.P. $\boldsymbol{a + a + d, a + 2d, a + 3d, \ldots}$

(v) The number of terms in the A.P when last term 'l' Given

$l = a + (n-1)\,d$ Gives $\boldsymbol{n = \left(\frac{l-a}{d}\right) + 1}$

(vi) 3 terms in A.P.

$\boldsymbol{a-d, a, a+d}$

(vii) 4 terms in A.P

$\boldsymbol{a-3d, a-d, a+d, a+3d}$

(viii) a, b, c are in A.P. if and only of

$\boldsymbol{b = \frac{a+c}{2}}$

$$\boxed{\boldsymbol{2b = a + c}}$$

Note:

- The common difference of an A.P. can be positive, negative or zero.
- An Arithmetic Progression having a common difference of zero is called a constant arithmatic progression.
- If every term is added or substracted by a constant, then the resulting seqeunce is also an A.P.
- If every term is multiplied or divided by a non-zero number, then the resulting sequence is also an A.P.

Type I: Basic problems based on A.P. (Verify A.P or not, Find A.P., Find unknown values)

Q.No. 1 (i) (ii) (iii) (iv) (v), Example 2.23 (i) (ii) (iii) 2 (i) (ii) (iii), Example 2.24, 3 (i) (ii), 8, 9

1. Check whether the following sequences are in A.P.

(i) $\boldsymbol{a-3, a-5, a-7, \ldots}$

$$t_2 - t_1 = (a-5) - (a-3)$$
$$= a - 5 - a + 3$$
$$= -2$$

$$t_3 - t_2 = (a-7) - (a-5)$$
$$= a - 7 - a + 5$$
$$= -2$$

Here $t_2 - t_1 = t_3 - t_2$

$\therefore$ Given sequence is an A.P.

(ii) $\boldsymbol{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots}$

$$t_2 - t_1 = \frac{1}{3} - \frac{1}{2}$$
$$= \frac{2-3}{6}$$
$$= \frac{-1}{6}$$
$$t_3 - t_2 = \frac{1}{4} - \frac{1}{3}$$
$$= \frac{3-4}{12}$$
$$= \frac{-1}{12}$$

Here $t_2 - t_1 \neq t_3 - t_2$

$\therefore$ Given sequence is not an A.P.

(iii) 9, 13, 17, 21, 25...

$$t_2 - t_1 = 13 - 9$$
$$= 4$$
$$t_3 - t_2 = 17 - 13$$
$$= 4$$

Here $t_2 - t_1 = t_3 - t_2$

Given sequence is an A.P.

(iv) $\boldsymbol{\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots}$

$$t_2 - t_1 = 0 - \left(\frac{-1}{3}\right)$$
$$= \frac{1}{3}$$

$$t_3 - t_2 = \frac{1}{3} - 0$$
$$= \frac{1}{3}$$

Here $t_2 - t_1 = t_3 - t_2$

$\therefore$ Given sequence is an A.P.

(v) 1, – 1, 1, – 1, 1, – 1, …

$$t_2 - t_1 = -1 - 1$$
$$= -2$$
$$t_3 - t_2 = 1 - (-1)$$
$$= 1 + 1$$
$$= 2$$

Here $t_2 - t_1 \neq t_3 - t_2$

$\therefore$ Given sequence is not an A.P.

Example 2.23

Check whether the following sequences are in A.P or not.

(i) $x + 2, 2x + 3, 3x + 4 \ldots$

$$t_2 - t_1 = (2x + 3) - (x + 2)$$
$$= 2x + 3 - x - 2$$
$$= x + 1$$
$$t_3 - t_2 = (3x + 4) - (2x + 3)$$
$$= 3x + 4 - 2x - 3$$
$$= x + 1$$

Here $t_2 - t_1 = t_3 - t_2$

Given sequence is an A.P.

(ii) 2, 4, 8, 16...

$$t_2 - t_1 = 4 - 2$$
$$= 2$$
$$t_3 - t_2 = 8 - 4$$
$$= 4$$

Here $t_2 - t_1 \neq t_3 - t_2$

Given sequence is not an A.P.

(iii) $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2} \ldots$

$$t_2 - t_1 = 5\sqrt{2} - 3\sqrt{2}$$
$$= 2\sqrt{2}$$
$$t_3 - t_2 = 7\sqrt{2} - 5\sqrt{2}$$
$$= 2\sqrt{2}$$

Here $t_2 - t_1 = t_3 - t_2$

Given sequence is an A.P.

---

2. First term 'a' and common difference 'd' are given below. Find the corresponding A.P.

(i) $a = 5, d = 6$

A.P

$a, a + d, a + 2d, a + 3d, \ldots$

$5, 5 + 6, 5 + 2\,(b), 5 + 3\,(6), \ldots$

$5, 11, 17, 23, \ldots$

(ii) $a = 7, d = -5$

A.P

$a + a + d, a + 2d, a + 3d, \ldots$

$7, 7 + (-5), 7 + 2\,(-5), 7 + 3\,(-5), \ldots$

$7, 7 - 5, 7 - 10, 7 - 15, \ldots$

$7, 2, -3, -8, \ldots$

(iii) $a = \frac{3}{4}, d = \frac{1}{2}$

A.P

$a, a + d, a + 2d, a + 3d, \ldots$

$$\frac{3}{4}, \frac{3}{4} + \frac{1}{2}, \frac{3}{4} + 2\left(\frac{1}{2}\right), \frac{3}{4} + 3\left(\frac{1}{2}\right), \ldots$$

$\frac{3}{4}, \frac{3+2}{4}, \frac{3+4}{4}, \frac{3+6}{4}, \ldots$

$\frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \ldots$

Example 2.24

Write an A.P. whose first term is 20 and common difference is 8.

$$a = 20$$

$$d = 8$$

A.P

$a + a + d, a + 2d, a + 3d, \ldots$

$20, 20 + 8, 20 + 2\,(8), 20 + 3\,(8), \ldots$

$20, 28, 36, 44, \ldots$

3. Find the first term and common difference of the Arithmetic progression whose n^{th} terms are given below.
(i) $t_n = -3 + 2n$ (ii) $t_n = 4 - 7n$

(i) $t_n = -3 + 2n$

Put $n = 1, 2, 3, 4, \ldots$

$t_1 = -3 + 2\,(1) = -3 + 2 = -1$

$t_2 = -3 + 2\,(2) = -3 + 4 = 1$

$t_3 = -3 + 2\,(3) = -3 + 6 = 3$

$t_4 = -3 + 2\,(4) = -3 + 8 = 5$

$\therefore$ **A.P** $-1, 1, 3, 5, \ldots$

$$a = -1$$

$$d = 3 - 1$$

$$= 2$$

(ii) $t_n = 4 - 7n$

Put $n = 1, 2, 3, 4, \ldots$

$t_1 = 4 - 7\,(1) = 4 - 7 = -3$

$t_2 = 4 - 7\,(2) = 4 - 14 = -10$

$t_3 = 4 - 7\,(3) = 4 - 21 = -17$

$t_4 = 4 - 7\,(4) = 4 - 28 = -24$

$\therefore$ **A.P** $-3, -10, -17, -24, \ldots$

$$a = -3$$

$$d = -10 + 3$$

$$= -7$$

8. If $3 + k, 18 - k, 5k + 1$ are in A.P then find k.

$3 + k, 18 - k, 5k + 1$ are in A.P.

$$\boxed{\therefore\ t_2 - t_1 = t_3 - t_2}$$

$$(18 - k) - (3 + k) = (5k + 1) - (18 - k)$$

$$18 - k - 3 - k = 5k + 1 - 18 + k$$

$$5 - 2k = 6k - 17$$

$$-2k - 6k = -17 - 15$$

$$-8k = -32$$

$$k = \frac{32}{8}$$

$$\boxed{k = 4}$$

9. Find x, y and z, Given that the numbers $x, 10, y, 24, z$ are in A.P.

$x, 10, y, 24, z$ are in A.P.

We know that in an A.P. common difference is a constant.

$$\therefore d = 10 - x \quad \ldots(1)$$

$$d = y - 10 \quad \ldots(2)$$

$$d = 24 - y \quad \ldots(3)$$

$$d = z - 24 \quad \ldots(4)$$

From (2) & (3)

$$y - 10 = 24 - y$$

$$y + y = 24 + 10$$

$$2y = 34$$

$$y = \frac{34}{2}$$

$$\boxed{y = 17}$$

$$\therefore (2) \Rightarrow d = y - 10$$
$$= 17 - 10$$

$$\boxed{d = 7}$$

$$(2) \Rightarrow d = 10 - x$$
$$7 = 10 - x$$
$$x = 10 - 7$$

$$\boxed{x = 3}$$

$$(4) \Rightarrow d = z - 24$$
$$7 = z - 24$$
$$7 + 24 = z$$

$$\boxed{31 = z}$$

$$\therefore x = 3$$
$$y = 17$$
$$z = 31$$

Type: II Problems based on n^{th} term $t_n = a + (n - d)\, d$, number of terms

Q.No.4, Example 2.25, 5, Example 2.26, 14, 6, 7, Example 2.17, 10, 12, Example 2.28

4. Find the 19^{th} term of an A.P. $-11, -15, -19, \ldots$.

$$a = -11$$
$$d = -15 - (-11)$$
$$= -15 + 11$$
$$= -4$$

$$\boxed{t_n = a + (n-1)\, d}$$

$$t_{19} = a + 18d$$
$$= -11 + 18(-4)$$
$$= -11 - 72$$
$$= -83$$

Example 2.25

Find the 15^{th}, 24^{th} and n^{th} term (general term) of an A.P. given 3, 15, 27, 39, …

A.P

3, 15, 27, 39, …

$$a = 3$$
$$d = 15 - 3 = 12$$

- $t_n = a + (n-1)\, d$

$$t_{15} = a + 14d$$
$$= 3 + 14(12)$$
$$= 3 + 168$$
$$= 171$$

- $t_{24} = a + 23d$

$$= 3 + 23(12)$$
$$= 3 + 276$$
$$= 279$$

- General term

$$t_n = a + (n-1)\, d$$
$$= 3 + (n-1)\, 12$$
$$= 3 + 12n - 12$$
$$= 12n - 9$$

---

5. Which term of an A.P. 16, 11, 6, 1, … is -54?

$$a = 16$$
$$d = 11 - 16$$
$$= -5$$
$$l = -54$$

$$\boxed{n = \left(\frac{l-a}{d}\right) + 1}$$

$$= \left(\frac{-54-16}{-5}\right) + 1 = \left(\frac{-70}{-5}\right) + 1$$
$$= 14 + 1$$
$$n = 15$$

$\therefore -54$ is 15^{th} term.

Example 2.26

Find the number of terms in the A.P. 3, 6, 9, 12, … 111.

$$a = 3$$
$$d = 6 - 3$$
$$= 3$$
$$l = 111$$
$$\boxed{n = \left(\frac{l-a}{d}\right) + 1}$$
$$= \left(\frac{111-3}{3}\right) + 1$$
$$= \frac{108}{3} + 1$$
$$= 36 + 1$$
$$n = 37$$

There are 37 terms in the A.P.

14. Priya earned Rs. 15,000 in the first month. Thereafter her salary increased by Rs. 1500 per year. Here expenses are Rs. 13,000 during the first year and the expenses increased by Rs. 900 per year. How long will it taker her to save Rs. 20,000 per month.

	Yearly salary Rs.	**Yearly expenses Rs.**	**Yearly savings Rs.**
Ist Year	15,000	13,000	2000
2nd Year	16,500	13,900	2600
3rd Year	18,000	14,800	3200

We have to find number of years taken to get yearly savings Rs. 20,000

From the table savings

$$a = 2000$$
$$d = 2600 - 2000$$
$$= 600$$
$$l = 20,000$$
$$n = \left(\frac{l-a}{d}\right) + 1$$
$$= \left(\frac{20,000 - 2000}{600}\right) + 1$$
$$= \frac{18,000}{600} + 1$$
$$= 30 + 1$$
$$n = 31 \text{ years}$$

6. Find the middle term(s) of an A.P. 9, 15, 21, 27, … 183.

$$a = 9$$
$$d = 15 - 9$$
$$= 6$$
$$l = 183$$
$$n = \left(\frac{l-a}{d}\right) + 1$$
$$= \left(\frac{183-9}{6}\right) + 1$$
$$= \frac{174}{6} + 1$$
$$= 29 + 1$$
$$\boxed{n = 30}$$

∴ There are 30 terms in the A.P.

$$n = 30 \text{ (even)}$$

∴ Middle terms are

$$\left(\frac{n}{2}, \frac{n}{2} + 1\right)^{th} \text{ terms}$$

i.e 15, 16th terms

- $t_{15} = a + 14d$

$$= 9 + 14\,(6)$$
$$= 9 + 84$$
$$= 93$$

- $t_{16} = a + 15d$

$$= 9 + 15\,(6)$$

$= 9 + 90$

$= 99$

$\therefore$ Middle terms are 93, 99

---

7. If nine times ninth term is equal to the fifteen times fifteenth term, show that six times twenty fourth term is zero.

Given

$$9t_9 = 15t_{15}$$

$$9(a + 8d) = 15(a + 14d)$$

$$9a + 72d = 15a + 210d$$

$$9a - 15a = 210d - 72d$$

$$-6a = 138d$$

$$a = \frac{138d}{-6}$$

$$\boxed{a = -23d}$$

To prove

$$6t_{24} = 0$$

LHS

$$6t_{24} = 6(a + 23d)$$

use $a = -23d$

$$= 6(-23d + 23d)$$

$$= 6(0)$$

$$6t_{24} = 0 \text{ RHS}$$

Hence proved.

Example 2.27

Determine the general term of an A.P. Whose 7th term is -1 and 16th term is 17.

Given

$$t_7 = -1$$

$$a + 6d = -1 \quad ...(1)$$

$$t_{16} = 17$$

$$a + 15d = 17 \quad ...(2)$$

Subtract (2) from (1)

$$a + 15d = 17$$
$$(-)\ (-)\ (+)$$
$$a + 6d = -1$$
$$9d = 18$$

$$d = \frac{18}{9}$$

$$\boxed{d = 2}$$

Put $d = 2$ in (2)

$$a + 15(2) = 17$$

$$a + 30 = 17$$

$$a = 17 - 30$$

$$\boxed{a = -13}$$

$\therefore$ General term

$$t_n = a + (n-1)d$$

$$= -13 + (n-1)(2)$$

$$= -13 + 2n - 2$$

$$t_n = 2n - 15$$

---

10. In a theatre, there are 20 scale in front row and 30 rows allotted. Each successive row contains two additional seats that its front row. How may seats are there in the last row?

$a = 20$

$d = 2$

$n = 30$

$$t_{30} = a + 29d$$

$$= 20 + 29(2)$$

$$= 20 + 58$$

$$= 78$$

$\therefore$ There are 78 seats in the last row.

12. The ratio of 6^{th} and 8^{th} term of an A.P. is 7:9, Find the ratio of 9^{th} term to 13^{th} term.

Given $\frac{t_6}{t_8} = \frac{7}{9}$

$$\frac{a+5d}{a+7d} = \frac{7}{9}$$

$$9(a+5d) = 7(a+7d)$$

$$9a + 45d = 7a + 49d$$

$$9a - 7a = 49d - 45d$$

$$2a = 4d$$

$$\boxed{a = 2d}$$

To find

$$\frac{t_9}{t_{13}} = \frac{a+8d}{a+12d}$$

Put $a = 2d$

$$= \frac{2d+8d}{2d+12d}$$

$$= \frac{10d}{14d}$$

$$= \frac{5}{7}$$

$\therefore$ ratio of 9^{th} term to 13^{th} term is 5:7.

Example 2.28

If l^{th}, m^{th} and n^{th} terms of an A.P are x, y, z respectively, than show that

(i) $\boldsymbol{x(m-n) + y(n-l) + z(l-m) = 0}$

(ii) $\boldsymbol{(x-y)n + (y-z)l + (z-x)m = 0}$

Given

$$t_l = x$$

$$a + (l-1)d = x \quad ...(1)$$

$$t_m = y$$

$$a + (m-1)d = y \quad (2)$$

$$t_n = z$$

$$a + (n-1)d = z \quad (3)$$

(i) LHS

$x(m-n) + y(n-l) + z(l-m)$

$= [a + (l-1)d](m-n) + [a + (m-1)d][n-l] + [a + (n-1)d](l-m)$

$= (a + ld - d)(m-n) + (a + md - d)(n-l) + (a + nd - d)(l-m)$

$= am + lmd - dm - na - nld + nd$
$\quad + na + mnd - dn - la - lmd + ld$
$\quad + al + nld - ld - ma - mnd + dm$

$= 0$ RHS

(ii) $x - y = a + (l-1)d] - [a + (m-1)d]$

$= a + ld - d - a - md + d$

$= ld - md$

$y - z = [a + (m-1)d] - [a + (n-1)d]$

$= a + md - d - a - nd + d$

$= md - nd$

$z - x = [a + (n-1)d] - [a + (l-1)d]$

$= a + nd - d - a - ld + d$

$= nd - ld$

LHS

$(x-y)n + (y-z)l + (z-x)m$

$= (ld - md)n + (md - nd)l + (nd - ld)m$

$= lnd - mnd + mld - nld + mnd - lmd$

$= 0$ RHS

Type II: 3 numbers, 4 numbers in an A.P. based sums

Q.No.11, Example 2.30, 2.29, 13

11. The sum of three consecutive terms that are in A.P is 27 and their product is 288. Find the three terms.

Let 3 numbers in A.P.

$$a-d, a, a+d$$

Given sum of 3 numbers $= 27$

$$a - d + a + a + d = 27$$
$$3a = 27$$
$$a = \frac{27}{3}$$
$$\boxed{a = 9}$$

Given product of 3 numbers = 288

$$(a - d)(a)(a + d) = 288$$
$$(a^2 - d^2)\,a = 288$$

Put $a = 9$

$$(9^2 - d^2)\,9 = 288$$
$$81 - d^2 = \frac{288}{9}$$
$$81 - d^3 = 32$$
$$-d^2 = 32 - 81$$
$$-d^2 = -49$$
$$\boxed{d = \pm 7}$$

$\therefore$ **3 numbers**

When $a = 9, d = 7$	When $a = 9, d = -7$
$a - d, a, a + d$	$a - d, a, a + d$
$9 - 7, 9, 9 + 7$	$9 + 7, 9, 9 - 7$
2, 9, 16	16, 9, 2

Example 2.30

A mother divides Rs. 207 into three parts such that the amount are in A.P. and gives it to her three children. The product of the least two amounts that the children has Rs. 4623. Find the amount received by each child.

Let the amount received by the three children be in the form of A.P. is given by

$a - d, a, a + d$

Total amount = Rs. 207

$$a - d + a + a + d = 207$$
$$3a = 207$$
$$a = \frac{207}{3}$$
$$\boxed{a = 69}$$

Given product of the least two amounts = 4623

$$a(a - d) = 4623$$
$$69(69 - d) = 4623$$
$$69 - d = \frac{4623}{69}$$
$$69 - d = 67$$
$$-d = 67 - 69$$
$$-d = -2$$
$$\boxed{d = 2}$$

$\therefore$ Amount Given by the mother to her three children are

$$a - d, a + a + d$$
$$69 - 2, 69, 69 + 2$$

Rs. 67, Rs. 69, Rs. 71

Example 2.29

In an A.P. sum of four consecutive terms is 28 and their sum of their squares is 276. Find the four numbers.

Let 4 terms in A.P.

$$a - 3d, a - d, a + d, a + 3d$$

Given

Sum = 28

$$a - 3d + a - d + a + d + a + 3d = 28$$
$$4a = 28$$
$$a = \frac{28}{4}$$
$$\boxed{a = 7}$$

Given

Sum of squares of 4 terms = 276

$$(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 276$$

$$a^2 - 6ad + 9d^2 + a^2 - 2ad + d^2 + a^2 + 2ad + d^2 + a^2 + 6ad + 9d^2 = 276$$

$$4a^2 + 20d^2 = 276$$

Put $\boldsymbol{a = 7}$

$$4\,(7)^2 + 20d^2 = 276$$

$$4\,(49) + 20d^2 = 276$$

$$196 + 20d^2 = 276$$

$$20d^2 = 276$$

$$20d^2 = 276 - 196$$

$$20d^2 = 80$$

$$d^2 = \frac{80}{20}$$

$$d^2 = 4$$

$$\boxed{d = \pm 2}$$

$\therefore$ **4 numbers**

$\boldsymbol{a = 7, d = 2}$	$\boldsymbol{a = 7, d = -2}$
$a - 3d, a - d, a + d, a + 3d$	
$7 - 6, 7 - 2, 7 + 2, 7 + 6$	$7 + 6, 7 + 2, 7 - 2, 7 - 6$
1, 5, 9, 13	13, 9, 5, 1

13. In a winter season let us take the temperature of Ooty from Monday to Friday to be in A.P. The sum of temperatures from Monday to Wednesday is 0°C and sum of the temperatures from Wednesday to Friday is 18°C. Find the temperature on each of the five days.

Let 5 consecutives days which is an A.P.

$$a, a + d, a + 2d, a + 3d, a + 4d$$

Given

$$a + a + d + a + 2d = 0$$

$$3a + 3d = 0$$

$$a + d = 0 \quad \text{...(1)}$$

Given $a + 2d + a + 3d + a + 4d = 18$

$$3a + 9d = 18$$

$$a + 3d = 6 \quad \text{...(2)}$$

$$a + 3d = 6$$

$$\overset{-}{a} + \overset{-}{d} = \overset{-}{0}$$

$$2d = 6$$

$$d = \frac{6}{2}$$

$$\boxed{d = 3}$$

Put $d = 3$ in (1)

$$a + 3 = 0$$

$$\boxed{a = -3}$$

$\therefore$ The temperature on each five days.

Monday $(a) = -3°\,C$

Tuesday $(a + d) = -3 + 3 = 0°\,C$

Wednesday $(a + 2d) = -3 + 6 = 3°\,C$

Thursday $(a + 3d) = -3 + 9 = 6°\,C$

Friday $(a + 4d) = -3 + 12 = 9°\,C$

Exercise 2.6

Key points

1. Series

The sum of the terms of a sequence is called series. Let $a_1, a_2, a_3, \ldots a_n \ldots$ be the sequences of real numbers.

Then the real number $a_1 + a_2 + a_3 + \ldots$ is defined as the series of real numbers.

2. Sum to '*n*' terms of an A.P

(i) Given first term 'a', common difference 'd' and number of terms 'n' Given

$$S_n = \frac{n}{2}\,[2a + (n - 1)\,d]$$

(ii) Given first term 'a' common difference 'd' and last term 'l' Given

$$S_n = \frac{n}{2}\,[a + l] \text{ where } n = \left(\frac{l - a}{d}\right) + 1$$

Type I Find S_n Given number of terms 'n'

Q.No.1. (i) (ii), Example 2.31, 3, 5, Example 2.34, 8, 10, 4, Example 2.35

1. Find the sum of the following

(i) 3, 7, 11, … upto 40 terms

(ii) 102, 97, 92, … up to 27 terms

(i) 3, 7, 11, … upto 40 terms

$a = 3$

$d = 7 - 3 = 4$

$n = 40$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

$$S_{40} = \frac{40}{2}[2(3) + (40-1)4]$$

$$= 20[6 + 156]$$

$$= 20 \times 162$$

$$S_{40} = 3240$$

(ii) 102, 97, 92, … up to 27 terms

$a = 102$

$d = 97 - 102$

$= -5$

$n = 27$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

$$S_{27} = \frac{27}{2}[2(102) + 26(-5)]$$

$$= \frac{27}{2}[204 - 150]$$

$$= \frac{27}{2} \times 54$$

$$= 27 \times 27$$

$$S_{27} = 729$$

Example 2.31

Find the sum of first 15 terms of the A.P. **$8, 7\frac{1}{4}, 6\frac{1}{2}, 5\frac{3}{4}, \ldots$**

$a = 8$

$$d = 7\frac{1}{4} - 8$$

$$= \frac{29}{4} - 8$$

$$= \frac{-3}{4}$$

$n = 15$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

$$S_{15} = \frac{15}{2}\left[2(8) + 14\left(\frac{-3}{4}\right)\right]$$

$$S_{15} = \frac{15}{2}\left[16 - \frac{21}{2}\right]$$

$$= \frac{15}{2} \times \frac{11}{2}$$

$$= \frac{165}{4}$$

3. Find the sum of first 28 terms of an A.P whose n^{th} term is $4n - 3$.

Given

$$t_n = 4n - 3$$

$$t_1 = 4(1) - 3 = 4 - 3 = 1$$

$$t_2 = 4(2) - 3 = 8 - 3 = 5$$

$$t_3 = 4(3) - 3 = 12 - 3 = 9$$

$\therefore$ **A.P** 1, 5, 9, …

$a = 1$

$d = 5 - 1$

$= 4$

$n = 28$ (Given)

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

$$S_{28} = \frac{28}{2}[2(1) + 27(4)]$$

$$= 14[2 + 108]$$

$$= 14 \times 110$$

$$S_{28} = 1540$$

5. The 104th term and 4th term of an A.P are 125 and 0. Find the sum of first 35 terms.

Given

$$t_{104} = 125$$

$$a + 103d = 125 \quad ...(1)$$

$$t_4 = 0$$

$$a + 3d = 0$$

$$a = -3d \quad ...(2)$$

Put (2) in (1)

$$-3d + 103d = 125$$

$$100d = 125$$

$$d = \frac{125}{100}$$

$$\boxed{d = \frac{5}{4}}$$

$$S_{15} = \frac{15}{2}\left[16 - \frac{21}{2}\right]$$

$$= \frac{15}{2} \times \frac{11}{2}$$

$$= \frac{165}{4}$$

3. Find the sum of first 28 terms of an A.P whose n^{th} term is $4n - 3$.

Given

$$t_n = 4n - 3$$

$$t_1 = 4(1) - 3 = 4 - 3 = 1$$

$$t_2 = 4(2) - 3 = 8 - 3 = 5$$

$$t_3 = 4(3) - 3 = 12 - 3 = 9$$

$\therefore$ **A.P** 1, 5, 9, ...

$$a = 1$$

$$d = 5 - 1$$

$$= 4$$

$$n = 28 \text{ (Given)}$$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

$$S_{28} = \frac{28}{2}[2(1) + 27(4)]$$

$$= 14[2 + 108]$$

$$= 14 \times 110$$

$$S_{28} = 1540$$

5. The 104th term and 4th term of an A.P are 125 and 0. Find the sum of first 35 terms.

Given

$$t_{104} = 125$$

$$a + 103d = 125 \quad ...(1)$$

$$t_4 = 0$$

$$a + 3d = 0$$

$$a = -3d \quad ...(2)$$

Put (2) in (1)

$$-3d + 103d = 125$$

$$100d = 125$$

$$d = \frac{125}{100}$$

$$\boxed{d = \frac{5}{4}}$$

$$(2) \Rightarrow a = -3\left(\frac{5}{4}\right)$$

$$\boxed{a = \frac{-15}{4}}$$

Given $n = 35$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

$$S_{35} = \frac{35}{2}\left[2\left(\frac{-15}{4}\right) + 34\left(\frac{5}{4}\right)\right]$$

$$= \frac{35}{2}\left[\frac{-30}{4} + \frac{170}{4}\right]$$

$$= \frac{35}{2} \times \frac{140}{4}$$

$$= \frac{1225}{2}$$

$$\boxed{S_{35} = 612.5}$$

Example 2.34

The 13th term of an A.P is 3 and the sum of first 13 terms is 234. Find the common difference and the sum of first 21 terms.

Given

$$t_{13} = 3$$

$$a + 12d = 3 \quad \ldots(1)$$

$$S_{13} = 234$$

$$\frac{13}{2}[2a + 12d] = 234$$

$$2a + 12d = 234 \times \frac{2}{13}$$

$$2a + 12d = 18 \times 2$$

$$2a + 12d = 36 \quad \ldots(2)$$

$$2a + 12d = 36$$

$$a + 12d = 3$$

$$a = 33$$

$$(1) \Rightarrow 33 + 12d = 3$$

$$12d = 3 - 33$$

$$12d = -30$$

$$d = \frac{-30}{12}$$

$$\boxed{d = \frac{-5}{2}}$$

Given $n = 21$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)\,d]}$$

$$S_{21} = \frac{21}{2}\left[2\,(33) + 20\left(\frac{-5}{2}\right)\right]$$

$$= \frac{21}{2}[66 - 50]$$

$$= \frac{21}{2} \times 6$$

$$= 21 \times 8$$

$$S_{21} = 168$$

8. Raghu wish to buy a laptop. He can buy it by paying Rs. 40,000 cash or by giving it in 10 installments as Rs. 4800 in the first month, Rs. 4750 in the second month, Rs. 4,700 in the third month and so on. If he pays the money in this fashion, find
(i) total amount paid in 10 installments
(ii) how much extra amount that he has to pay than the cost?

Given

First month, 2nd month, 3rd month installments.

4800, 4750, 4700, … form an A.P

$$a = 4800$$

$$d = 4750 - 4800$$

$$= -50$$

$n = 10$ (installments)

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)\,d]}$$

$$S_{10} = \frac{10}{2}[2\,(4800) + 9\,(-50)]$$

$$= 5\,[9600 - 450]$$

$$= 5 \times 9150$$

$$= 45{,}750$$

∴ (i) Total amount paid in 10 installments is Rs. 45,750

(ii) 45,750 − 40,000 = Rs. 5,750 is the extra amount

10. A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two bricks less than the previous step.
(i) How many bricks are required for the top most step?
(ii) How many bricks are required to build the staircase?

Given **A.P:** 100, 98, 96, …

$$n = 30 \text{ steps}$$
$$a = 100 \text{ bricks}$$
$$d = -2$$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

$$S_{30} = \frac{30}{2}[2(100) + 29(-2)]$$
$$= 15[200 - 58]$$
$$= 15 \times 142$$
$$= 2130$$

$$t_{30} = a + 29d$$
$$= 100 + 29(-2)$$
$$= 100 - 58 = 42$$

(i) 42 bricks are required for the top most step?

(ii) 2130 brick are required to build the stair case?

---

4. The sum of first n terms of a certain series is given as $2n^2 - 3n$. Show that the series is an A.P.

Given

$$S_n = 2n^2 - 3n$$

- $S_1 = 2(1)^2 - 3(1)$

$$= 2 - 3$$
$$S_1 = -1$$
$$\boxed{\therefore t_1 = -1}$$

- $S_2 = 2(2)^2 - 3(2)$

$$= 8 - 6$$
$$= 2$$
$$S_2 = 2$$
$$t_1 + t_2 = 2$$
$$- + t_2 = 2$$
$$t_2 = 2 + 1$$
$$\boxed{t_2 = 3}$$

- $S_3 = 2(3)^2 - 3(3)$

$$= 18 - 9$$
$$S_3 = 9$$
$$t_1 + 4_2 + t_3 - 9$$
$$-1 + 3 + t_3 = 9$$
$$t_3 = 9 + 1 - 3$$
$$\boxed{t_3 = 7}$$

$\therefore -1, 3, 7, \ldots$ is an A.P since $d = 4$

Example 2.35

In a A.P the sum of first n term is $\frac{5n^2}{2} + \frac{3n}{2}$. Find the 17th term.

The 17th term can be obtained by subtracting the sum of first 16 terms from the sum of first 17 terms.

$$S_n = \frac{5n^2}{2} + \frac{3n}{2}$$

- $S_{17} = \frac{5(17)^2}{2} + \frac{3(17)}{2}$

$$= \frac{1445}{2} - \frac{51}{2}$$
$$= \frac{1496}{2}$$
$$= 748$$

- $S_{16} = \frac{5\,(16)^2}{2} + \frac{3\,(16)}{2}$

$$= \frac{1280}{2} + \frac{48}{2}$$

$$= \frac{1328}{2}$$

$$= 664$$

$\therefore\ t_{17} = S_{17} - S_{16}$

$$= 748 - 664$$

$$\boxed{t_{17} = 84}$$

Type: II Find '*in*' Given S_n

Q.No.2, 9, Example 2.33

2. How may consecutive odd integers beginning with 5 will sum to 480?

Given data

A.P $5 + 7 + 9 + \ldots$

$$S_n = 480$$

$$a = 5$$

$$d = 7 - 5$$

$$= 2$$

$$S_n = 480$$

$$n = ?$$

$$\boxed{S_n = \frac{n}{2}\,[2a + (n-1)\,d]}$$

$$480 = \frac{n}{2}\,[2\,(5) + (n-1)\,2]$$

$$480 \times 2 = n\,[10 + 2n - 2]$$

$$960 = n\,[8 + 2n]$$

$$960 = 8n + 2n^2$$

$$2n^2 + 8n - 960 = 0$$

$$n^2 + 4n - 480 = 0$$

$$(n + 24)\,(n - 20) = 0$$

$n + 24 = 0$ | $n - 20 = 0$

$n = -24$ | $n = 20$

$n = -24$ not is not possible

$$\therefore\ n = 20$$

9. A man repays a loan of Rs. 65,000 by paying Rs. 400 in the first month and then increasing the payment by Rs. 300 every month. How long will it take for him to clear the loan?

Given

$$S_n = \text{Rs. } 65{,}000$$

$$a = \text{Rs. } 400$$

$$d = \text{Rs. } 300$$

$$n = ?$$

$$\boxed{S_n = \frac{n}{2}\,[2a + (n-1)\,d]}$$

$$65{,}000 = \frac{n}{2}\,[2\,(400) + (n-1)\,300]$$

$$65{,}000 \times 2 = n\,[800 + 300n - 300]$$

$$130000 = n\,[500 + 300n]$$

$$130000 = 500n + 300n^2$$

$$300n^2 + 500n - 130000 = 0$$

$$3n^2 + 5n - 1300 = 0$$

$$(n - 20)\left(n + \frac{65}{3}\right) = 0$$

$n - 20 = 0$ | $n + \frac{65}{3} = 0$

$n = 20$ | $n = \frac{-65}{3}$

not possible

-3900

-60 65

$\frac{-60}{3}$, $\frac{65}{3}$

-20, $\frac{65}{3}$

$\therefore$ He will take 20 months to clear the loan.

Example 2.33

How many terms of the series $1 + 5 + 9 + \ldots$ must be taken so that their sum is 190?

$$a = 1$$

$$d = 5 - 1$$

$$= 4$$

$$S_n = 190$$

$$n = ?$$

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

$$190 = \frac{n}{2}[2(1) + (n-1)4]$$

$$190 \times 2 = n[2 + 4n - 4]$$

$$380 = n[4n - 2]$$

$$380 = 4n^2 - 2n$$

$$4n^2 - 2n - 380 = 0$$

$$2n^2 - n - 190 = 0$$

-380

$-20 \quad 19$

$\frac{-20}{2}, \quad \frac{19}{2}$

$-10, \quad \frac{19}{2}$

$$(n-10)\left(n + \frac{19}{2}\right) = 0$$

$n - 10 = 0$	$n + \frac{19}{2} = 0$
$n = 10$	$n = \frac{-19}{2}$
	not possible

Type III Find S_n Given last term

Q.No.1. (iii) Example 2.32, 6, 7, Example 2.36, 2.37, 2.38

1. Find the sum of the following

(iii) $6 + 13 + 20 + \ldots + 97$

$$a = 6$$

$$d = 13 - 6 = 7$$

$$l = 97$$

$$n = \left(\frac{l-a}{d}\right) + 1$$

$$= \left(\frac{97-6}{7}\right) + 1$$

$$= \frac{91}{7} + 1$$

$$= 13 + 1$$

$$\boxed{n = 14}$$

- $S_n = \frac{n}{2}[a + l]$

$$S_{14} = \frac{14}{2}[6 + 97]$$

$$= 7 \times 103$$

$$S_{14} = 721$$

Example 2.32

***Find the sum of* $0.40 + 0.43 + 0.46 + \ldots + 1$.**

$$a = 0.40$$

$$d = 0.43 - 0.40 = 0.03$$

$$l = 1$$

- $n = \left(\frac{l-a}{d}\right) + 1$

$$= \left(\frac{1 - 0.40}{0.03}\right) + 1$$

$$= \left(\frac{0.60}{0.03}\right) + 1$$

$$= 20 + 1$$

$$n = 21$$

- $S_n = \frac{n}{2}[a + (1)]$

$$S_{21} = \frac{21}{2}[0.40 + 1]$$

$$= \frac{21}{2} \times 1.40$$

$$= 21 \times 0.70$$

$$S_{21} = 14.7$$

6. Find the sum of all odd positive integers less than 450.

Given data

A.P

$1 + 3 + 5 + \ldots + 449$

$$a = 1$$

$$d = 3 - 1 = 2$$

$$l = 449$$

- $n = \left(\dfrac{l-a}{d}\right) + 1$

$$= \left(\frac{449-1}{2}\right) + 1$$

$$= \frac{448}{2} + 1$$

$$= 224 + 1$$

$$\boxed{n = 225}$$

- $S_n = \dfrac{n}{2}[a + l]$

$$= \frac{225}{2}[1 + 449]$$

$$= \frac{225 \times 450}{2}$$

$$= 225 \times 225$$

$$S_{225} = 50625$$

Note

- Sum of first 'n' odd numbers

$$S_n = n^2$$

---

7. Find the sum of all natural numbers between 602 and 902 which are not divisible by 4.

- First we have to find sum of all numbers between 602 and 902

$$603 + 604 + \ldots + 901$$

$$n = 901 - 602$$

$$= 299$$

$$S_n = \frac{n}{2}[a + l]$$

$$= \frac{299}{2}(603 + 901)$$

$$= \frac{299}{2} \times 1504$$

$$= 299 \times 752$$

$$S_{301} = 2{,}24{,}848$$

- Now sum of all numbers divisible by 4 between 602 and 902

$$a = 602 + 2$$

$$= 604$$

$$l = 902 - 2$$

$$= 900$$

$$d = 4$$

$$\begin{array}{r|l} & 15 \\ \hline 4 & 602 \\ & 4 \\ \hline & 20 \\ & 20 \\ \hline & 2 \end{array}$$

$$\begin{array}{r|l} & 225 \\ \hline 4 & 902 \\ & 8 \\ \hline & 10 \\ & 8 \\ \hline & 22 \\ & 20 \\ \hline & 2 \end{array}$$

$$\therefore n = \left(\frac{l-a}{d}\right) + 1$$

$$= \left(\frac{900-604}{4}\right) + 1$$

$$= \frac{296}{4} + 1$$

$$= 74 + 1$$

$$\boxed{n = 75}$$

- $S_n = \dfrac{n}{2}[a + l]$

$$= \frac{75}{2}[604 + 900]$$

$$= \frac{75}{2} \times 1504$$

$$= 75 \times 752$$

$$S_{75} = 56{,}400$$

$\therefore$ Sum of all numbers between 602 and 902 which are not divisible by 4 is

$$2{,}24{,}845 - 56{,}400 = 1{,}68{,}448$$

Example 2.36

Find the sum of all natural numbers between 300 and 600 which are divisible by 7

First number, last number divisible by 7 between 300 and 600 are

$$301 + 308 + 315 + \ldots + 595$$

$$a = 301$$

$$d = 7$$

$$l = 595$$

$$n = \left(\frac{l-a}{d}\right) + 1$$

$$= \left(\frac{595-301}{7}\right) + 1$$

$$= \frac{294}{7} + 1$$

$$= 42 + 1 = 43$$

$$S_n = \frac{n}{2}[a+l]$$

$$= \frac{43}{2}[301 + 595]$$

$$= \frac{43}{2} \times 896$$

$$= 43 \times 448$$

$$S_{43} = 19{,}264$$

$$\begin{array}{r|l} & 4 \\ \hline 7 & 300 \\ & 28 \\ \hline & 20+1 \\ & \quad =21 \end{array}$$

$300 + 1 = 301$

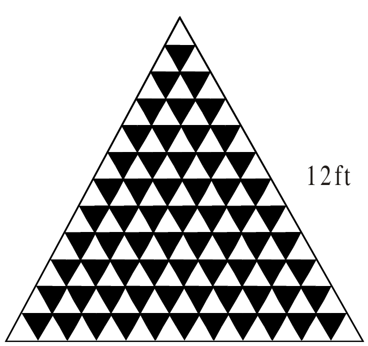
$$\begin{array}{r|l} & 85 \\ \hline 7 & 600 \\ & 56 \\ \hline & 40 \\ & 35 \\ \hline & 5 \end{array}$$

$600 - 5 = 595$

Example 2.37

A mosaic is designed in the shape of an equilateral triangle, 12 ft on each side. Each tile in the mosaic is in the shape of an equilateral triangle of 12 inch side. The tiles are alternate in colour as shown in the figure. Find the number of tiles of each colour and total number of tiles in the mosaic.

Since the mosaic is in the shape of the equilateral triangle of 12 ft, and the tile is in shape of an equilateral triangle of 12 inch (1 ft), there will be 12 rows in the mosaic.



- From the figure, number of white tiles in each row are 1, 2, 3, 4, … 12 which clearly forms an A.P.

$\therefore$ Number of white tiles $= \frac{n}{2}(a+l)$

$$= \frac{12}{2}(1+12)$$

$$= 6 \times 13$$

$$= 78$$

- Number of blue tiles in each row are 0, 1, 2, 3, … 11 which is also an A.P.

$\therefore$ Number of blue tiles $= \frac{n}{2}[a+l]$

$$= \frac{12}{2}(0+11)$$

$$= 6 \times 11$$

$$= 66$$

$\therefore$ The total number of tiles in the mosaic

$$= 78 + 66$$

$$= 144$$

Example 2.38

The houses of a street are numbered from 1 to 49. Senthil's house is numbered such that the sum of numbers of the houses prior to senthil's house is equal to the sum of numbers of the houses following senthil's house. Find senthil's house number.

Let senthil's house number be x

It is given that

$$1 + 2 + 3 + \ldots + (x-1) = (x+1) + (x+2) + \ldots + 49$$

$$1 + 2 + 3 + \ldots + (x-1) = (1 + 2 + 3 + \ldots + 49) - (1 + 2 + 3 + \ldots + x)$$

Use $S_n = \frac{n}{2}(a+l)$

$$\frac{x-1}{2}[1 + (x-1)] = \frac{49}{2}(1+49) - \frac{x}{2}(1+x)$$

$$\frac{x-1}{2}(x) = \frac{49}{2} \times 50 - \frac{x-x^2}{2}$$

$$\frac{x^2-x}{2} = \frac{1}{2}[49 \times 50 - x - x^2]$$

$$x^2 - x = 2450 - x - x^2$$

$$x^2 + x^2 = 2450$$

$$2x^2 = \frac{2450}{2}$$

$$x^2 = 1225$$

$$x = 35$$

$\therefore$ Senthil's house number is 35.

Type IV Prove the following
Q.No. 11, 12, Example 2.39

11. If $S_1, S_2, S_3, \dots S_m$ are the sums of n terms of m A.P's whose first terms are $1, 2, 3, \dots m$ and whose common differences are $1, 3, 5, \dots (2m-1)$ respectively, then show that $S_1 + S_2 + S_3 + \dots + S_m = \frac{1}{2} mn\,(mn+1)$

- First term $a = 1, d = 1$

$$S_1 \Rightarrow S_n = \frac{n}{2}[2a + (n-1)\,d]$$

$$= \frac{n}{2}[2\,(1) + (n-1)\,1]$$

$$= \frac{n}{2}[2 + n - 1]$$

$$= \frac{n}{2}(n+1)$$

- First term $a = 2, d = 3$

$$S_2 \Rightarrow S_n = \frac{n}{2}[2a + (n-1)\,d]$$

$$= \frac{n}{2}[2\,(2) + (n-1)\,3]$$

$$= \frac{n}{2}[4 + 3n - 3]$$

$$= \frac{n}{2}[3n + 1]$$

- First term $a = 3, d = 5$

$$S_3 \Rightarrow S_n = \frac{n}{2}[2a + (n-1)\,d]$$

$$= \frac{n}{2}[2\,(3) + (n-1)\,5]$$

$$= \frac{n}{2}[6 + 5n - 5]$$

$$= \frac{n}{2}[5n + 1]$$

- First term $a = m, d = (2m-1)$

$$S_m \Rightarrow S_n = \frac{n}{2}[2a + (n-1)\,d]$$

$$= \frac{n}{2}[2\,(m) + (n-1)\,(2m-1)]$$

$$= \frac{n}{2}[2m + 2mn - 2m - n + 1]$$

$$S_m = \frac{n}{2}[2mn - n + 1]$$

$$\therefore S_1 + S_2 + S_3 + \dots + S_m$$

$$= \frac{n}{2}(n+1) + \frac{n}{2}(3n+1) + \frac{n}{2}[5n+1] + \dots + \frac{n}{2}[2mn - n + 1]$$

$$= \frac{n}{2}[n + 1 + 3n + 1 + 5n + 1 + \dots + 2mn - n + 1]$$

$$= \frac{n}{2}[n\,(1 + 3 + 5 + \dots + (2m-1)] + [1 + 1 + 1 + \dots m \text{ terms}]$$

$$= \frac{n}{2}[n \times m^2 + m]$$

Sum of first n odd numbers is n^2

$$= \frac{1}{2}[n^2 m^2 + mn]$$

$$= \frac{1}{2} mn\,(mn+1) \text{ proved}$$

---

12. Find the sum $\left[\frac{a-b}{a+b} + \frac{3a-2b}{a+b} + \frac{5a-3b}{a+b} + \dots \text{ to 12 terms}\right]$

$$\frac{1}{a+b}[(a-b) + (3a-2b) + (5a-3b) + \dots \text{ to 12 terms}]$$

Here $a = a - b$

$$d = (3a - 2b) - (a - b)$$

$= 3a - 2b - a + b$

$= 2a - b$

$n = 12$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore = \frac{1}{a+b} \times \frac{12}{2}[2(a-b) + 11(2a-b)]$$

$$= \frac{6}{a+b}[2a - 2b + 22a - 11b]$$

$$= \frac{6}{a+b}[24a - 13b]$$

Example 2.39

The sum of first $n, 2n$ and $3n$ terms of an A.P are S_1, S_2 and S_3 respectively. Prove that $S_3 = 3(S_2 - S_1)$.

Given

$$S_1 = \frac{n}{2}[2a + (n-1)d]$$

$$S_2 = \frac{2n}{2}[2a + (2n-1)d]$$

$$S_3 = \frac{3n}{2}[2a + (3n-1)d]$$

Let

$$S_2 - S_1 = \frac{2n}{2}[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{2n}{2}[2a + 2nd - d] - \frac{n}{2}[2a + nd - d]$$

$$= \frac{n}{2}[4a + 4nd - 2d - 2a - nd + d]$$

$$= \frac{n}{2}[2a + 3nd - d]$$

$$= \frac{n}{2}[2a + (3n-1)d]$$

$$\therefore 3(S_2 - S_1) = \frac{3n}{2}[2a + (3n-1)d]$$

$= S_3$ Hence proved

Exercise 2.7

Key points

Geometric Progression (G.P)

A Geometric progression is a sequence in which each term is obtained by multiplying a fixed non-zero number to the preceding term except the first term.

The fixed number is called common ratio. It is denoted by 'r'.

(i) General form of G.P

$$a, ar, ar^3, \ldots ar^{n-1}, \ldots$$

(ii) First term is denoted by 'a' and common ratio is 'r' where $r = \frac{t_2}{t_1}$

(iii) To verify Given sequence is G.P or not

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$

(iv) n^{th} term (or) General term

$$\boxed{t_n = ar^{n-1}}$$

(v) Three consecutive terms in G.P

$$\frac{a}{r}, a, ar$$

(vi) Four consecutive terms in G.P.

$$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$$

(vii) When each term of a G.P is multiplied or divided by a non-zero constant then the resulting sequence is also a G.P.

(viii) Three non-zero number a, b, c are in G.P. if and only if $\boxed{b^2 = ac}$

Type I Basic sums, To verify G.P or not, write the first three terms.

Q.No. 1. (i) to (vii), Example 2.40 (i) (ii) (iii), 2 (i) (ii) (iii), Example 2.41, 4

1. Which of the following sequences are in G.P?

(i) 3, 9, 27, 81, …

To check if a given sequence form a G.P. We have to see if the ratio between successive terms are equal.

- $\frac{t_2}{t_1} = \frac{9}{3} = 3$
- $\frac{t_3}{t_2} = \frac{27}{9} = 3$

Here $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

Hence Given Sequence is a G.P.

(ii) 4, 44, 444, 4444, …

- $\frac{t_2}{t_1} = \frac{44}{4} = 11$
- $\frac{t_3}{t_2} = \frac{444}{44} = \frac{111}{11}$

Here $\frac{t_2}{t_1} \neq \frac{t_3}{t_2}$

$\therefore$ Given sequence is not a G.P.

(iii) 0.5, 0.05, 0.005, …

- $\frac{t_2}{t_1} = \frac{0.05}{0.5} = \frac{0.5}{5} = 0.1$
- $\frac{t_3}{t_2} = \frac{0.005}{0.05} = \frac{0.5}{5} = 0.1$

Here $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

$\therefore$ Given sequence is a G.P.

(iv) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \ldots$

- $\frac{t_2}{t_1} = \frac{1}{6} \times \frac{3}{1} = \frac{1}{2}$
- $\frac{t_3}{t_2} = \frac{1}{12} \times \frac{6}{1} = \frac{1}{2}$

Here $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

$\therefore$ Given sequence is a G.P.

(v) 1, – 5, 25, – 125, …

- $\frac{t_2}{t_1} = \frac{-5}{1} = -5$
- $\frac{t_3}{t_2} = \frac{25}{-5} = -5$

Here $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

Given sequence is a G.P.

(vi) 120, 60, 30, 18, …

- $\frac{t_2}{t_1} = \frac{60}{120} = \frac{1}{2}$
- $\frac{t_3}{t_2} = \frac{30}{60} = \frac{1}{2}$
- $\frac{t_4}{t_3} = \frac{18}{30} = \frac{3}{5}$

Here $\frac{t_2}{t_1} \neq \frac{t_4}{t_3}$

Given sequence is not a G.P.

(vii) 16, 4, 1, $\frac{1}{4}$ …

- $\frac{t_2}{t_1} = \frac{4}{16} = \frac{1}{4}$
- $\frac{t_3}{t_2} = \frac{1}{4}$

Here $\frac{t_2}{t_1} = \frac{t_3}{t_2}$

Given sequence is a G.P.

Example 2.40

Which of the following sequences form a Geometric progression?

(i) 7, 14, 21, 28, …

- $\frac{t_2}{t_1} = \frac{14}{7} = 2$
- $\frac{t_3}{t_2} = \frac{21}{14} = \frac{3}{2}$

Here $\frac{t_2}{t_1} \neq \frac{t_3}{t_2}$

$\therefore$ Given sequence is not a G.P.

(ii) $\frac{1}{2}$, 1, 2, 4, …

- $\frac{t_2}{t_1} = \frac{1}{1/2} = 2$
- $\frac{t_3}{t_5} = \frac{2}{1} = 2$
- $\frac{t_4}{t_5} = \frac{4}{2} = 2$

$\therefore$ Given sequence is a G.P.

(iii) 5, 25, 50, 75, …

- $\frac{t_2}{t_1} = \frac{25}{5} = 5$
- $\frac{t_3}{t_2} = \frac{50}{25} = 2$
- $\frac{t_4}{t_3} = \frac{75}{50} = \frac{3}{2}$

Since the ratios between successive terms are not equal.

Hence Given sequence is not a G.P.

2. Write the first three terms of the G.P. whose first term and the common ratio are given below.

(i) $a = 6, r = 3$

G.P a, ar, ar^2 …

First term $'a' = 6$

Second term $ar = 6 \times 3 = 18$

Third term $ar^2 = 6\,(3)^2 = 54$

(ii) $a = \sqrt{2}, r = \sqrt{2}$

$t_1 = a = \sqrt{2}$

$t_2 = ar\;(\sqrt{2})\;(\sqrt{2}) = 2$

$t_3 = ar^2 = (\sqrt{2})\;(\sqrt{2})^2 = 2\sqrt{2}$

(iii) $a = 1000, r = \frac{2}{5}$

$t_1 = a = 1000$

$t_2 = ar = 1000 \times \frac{2}{5} = 400$

$t_3 = ar^2 = 1000 \times \frac{2}{5} \times \frac{2}{5} = 160$

Example 2.41

Find the geometric progression whose first term and common ratios are given by

(i) **$a = -7, r = 6$**

(ii) **$a = 256, r = 0.5$**

(i) $a = -7, r = 6$

$t_1 = a = -7$

$t_2 = ar = (-7)\,(6) = -42$

$t_3 = ar^2 = (-7)\,(6)\,(6) = -252$

$\therefore$ **G.P** $-7, -42, -252$

(ii) $a = 256, r = 0.5$

$a = 256,\ r = \frac{1}{2}$

$$t_1 = a = 256$$

$$t_2 = ar = 256 \times \frac{1}{2} = 128$$

$$t_3 = ar^2 = 256 \times \frac{1}{2} \times \frac{1}{2} = 64$$

G.P 256, 128, 64, …

---

4. Find x so that $x + 6, x + 12$ and $x + 15$ are consecutive terms of a Geometric progression.

$x + 6, x + 12, x + 15$ are in G.P.

$$\boxed{a, b, c \text{ are in G.P then } b^2 = ac}$$

$$(x + 12)^2 = (x + 6)(x + 15)$$

$$x^2 + 24x + 144 = x^2 + 21x + 90$$

$$24x - 21x = 90 - 144$$

$$3x = -54$$

$$x = \frac{-54}{3}$$

$$\boxed{x = -18}$$

Type II: $t_n = ar^{n-1}$ based problems

Q.No: 3, Example 2.42, 5 (i) (ii), 6, 7, Example 2.43.

3. In a G.P. 729, 243, 81, … find t_7

$$a = 729$$

$$r = \frac{t_2}{t_1} = \frac{243}{729} = \frac{1}{3}$$

$$\boxed{t_n = ar^{n-1}}$$

$$t_7 = ar^6$$

$$= 729 \times \left(\frac{1}{3}\right)^6$$

$$= 729 \times \frac{1}{729}$$

$$= 1$$

Example 2.42

Find the 8^{th} term of the G.P. 9, 3, 1, …

$$a = 9$$

$$r = \frac{t_2}{t_1} = \frac{3}{9} = \frac{1}{3}$$

$$\boxed{t_n = ar^{n-1}}$$

$$t_8 = ar^7$$

$$= 9\left(\frac{1}{3}\right)^7$$

$$= \frac{1}{3^5}$$

$$t_8 = \frac{1}{243}$$

---

5. Find the number of terms in the following G.P.

(i) 4, 8, 16, …, 8192?

$$a = 4$$

$$r = \frac{8}{4} = 2$$

$$t_n = 8192$$

$$\boxed{t_n = ar^{n-1}}$$

$$8192 = 4\,(2)^{n-1}$$

$$\frac{8192}{4} = 2^{n-1}$$

$$2048 = 2^{n-1}$$

$$2^{11} = 2^{n-1}$$

$$11 + 1 = n$$

$$12 = n$$

2	2048
2	1024
2	512
2	256
2	128
2	64
2	32
2	16
2	8
2	4
	2

∴ The number of terms is 12.

(ii) $\mathbf{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \frac{1}{2187}}$

$$a = \frac{1}{3}$$

$$r = \frac{t_2}{t_1} = \frac{1}{9} \times \frac{3}{1} = \frac{1}{3}$$

$$t_n = \frac{1}{2187}$$

$$\boxed{t_n = ar^{n-1}}$$

$$\frac{1}{2187} = \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$$

$$\frac{3}{2187} = \left(\frac{1}{3}\right)^{n-1}$$

$$\frac{1}{729} = \left(\frac{1}{3}\right)^{n-1}$$

$$\left(\frac{1}{3}\right)^{6} = \left(\frac{1}{3}\right)^{n-1}$$

$$6 = n - 1$$

$$6 + 1 = n$$

$$7 = n$$

$\therefore$ The number of terms is 7.

---

6. In a G.P. the 9$^{\text{th}}$ term is 32805 and 6$^{\text{th}}$ term is 1215. Find the 12$^{\text{th}}$ term.

Given

$t_9 = 32805$ $\quad$ $t_6 = 1215$

$ar^8 = 32085 \ldots (1)$ $\quad$ $ar^5 = 1215 \ldots (2)$

$$(1) \div (2) \quad \frac{ar^8}{ar^5} = \frac{32805}{1215}$$

$$r^3 = 27$$

$$\boxed{r = 3}$$

$$(2) \Rightarrow \ a\,(3)^5 = 1215$$

$$243a = 1215$$

$$a = \frac{1215}{243}$$

$$\boxed{a = 5}$$

$$\therefore t_n = ar^{n-1}$$

$$t_{12} = ar^{11}$$

$$= 5\,(3)^{11}$$

---

7. Find the 10$^{\text{th}}$ term of a G.P whose 8$^{\text{th}}$ term is 768 and the common ratio is 2.

Given

$$t_8 = 768 \text{ and } r = 2$$

$$ar^7 = 768$$

$$a\,(2)^7 = 768$$

$$128a = 768$$

$$a = \frac{768}{128}$$

$$\boxed{a = 6}$$

$$\therefore t_n = ar^{n-1}$$

$$t_{10} = ar^9$$

$$= 6\,(2)^9$$

$$= 6 \times 512$$

$$t_{10} = 3072$$

Example 2.43

In a Geometric progression the 4$^{\text{th}}$ term is $\mathbf{\frac{8}{9}}$ and the 7$^{\text{th}}$ term is $\mathbf{\frac{64}{243}}$. Find the Geometric progression.

Given

$t_4 = \frac{8}{9}$ $\quad$ $t_7 = \frac{64}{243}$

$ar^3 = \frac{8}{9} \ldots (1)$ $\quad$ $ar^6 = \frac{64}{243} \ldots (2)$

$(2) \div (1)$

$$\frac{ar^6}{ar^3} = \frac{64}{243} \times \frac{9}{8}$$

$$r^3 = \frac{8}{27}$$

$$r^3 = \left(\frac{2}{3}\right)^3$$

$$\boxed{\boldsymbol{r = \frac{2}{3}}}$$

Put $r = \frac{2}{3}$ in (1)

$$a\left(\frac{2}{3}\right)^3 = \frac{8}{9}$$

$$\frac{8a}{27} = \frac{8}{9}$$

$$a = \frac{8}{9} \times \frac{27}{8}$$

$$\boxed{\boldsymbol{a = 3}}$$

$\therefore$ **G.P**

$$a, ar, ar^2, \ldots$$

$$3, 3\left(\frac{2}{3}\right), 3\left(\frac{4}{9}\right), \ldots$$

$$3, 2, \frac{4}{3}, \ldots$$

Type III: 3 numbers $\frac{a}{r}$, a, ar in G.P. based problems

Q.No: 9, Example 2.44

9. In a G.P the product of three consecutive terms is 27 and the sum of the product of two terms taken at a time is $\frac{57}{2}$. Find three terms.

Let 3 consecutive terms in G.P.

$$\frac{a}{r}, a, ar$$

Given

Product of 3 terms $= 27$

$$\frac{a}{r} \times a \times ar = 27$$

$$a^3 = 27$$

$$\boxed{\boldsymbol{a = 3}}$$

Given

Sum of the product of two terms taken at a time $= \frac{57}{2}$

$$\left(\frac{a}{r} \times a\right) + (a \times ar) + \left(ar \times \frac{a}{r}\right) = \frac{57}{2}$$

$$\frac{a^2}{r} + a^2 r + a^2 = \frac{57}{2}$$

$$a^2\left(\frac{1}{r} + r + 1\right) = \frac{57}{2}$$

$$\frac{1}{r} + r + 1 = \frac{57}{2a^2}$$

Put $a = 3$

$$\frac{1 + r^2 + r}{r} = \frac{57}{2 \times 9}$$

$$\frac{1 + r^2 + r}{r} = \frac{19}{6}$$

$$6 + 6r^2 + 6r = 19r$$

$$6r^2 + 6r - 19r + 6 = 0$$

$$6r^2 - 13r + 6 = 0$$

36

$-4 \qquad -9$

$\frac{-4}{6}, \qquad \frac{-9}{6}$

$\frac{-2}{3}, \qquad \frac{-3}{2}$

$$\left(r - \frac{2}{3}\right)\left(r - \frac{3}{2}\right) = 0$$

$$r - \frac{2}{3} = 0 \qquad r - \frac{3}{2} = 0$$

$$\boldsymbol{r = \frac{2}{3}} \qquad \boldsymbol{r = \frac{3}{2}}$$

$\therefore$ **3 terms**

$a = 3, r = \frac{2}{3}$	$a = 3, r = \frac{3}{2}$
$\frac{a}{r}, a, ar$	
$\frac{3}{2/3}, 3, 3 \times \frac{2}{3}$	$\frac{3}{3/2}, 3, 3 \times \frac{3}{2}$
$\boldsymbol{\frac{9}{2}, 3, 2}$	$\boldsymbol{2, 3, \frac{9}{2}}$

Example 2.44

The product of three consecutive terms of a Geometric progression is 343 and their sum is $\frac{91}{3}$. Find the three terms.

Let

3 terms in G.P.

$$\frac{a}{r}, a, ar$$

Given

$$\text{Product of 3 terms} = 343$$

$$\frac{a}{r} \times a \times ar = 343$$

$$a^3 = 343$$

$$a^3 = 7^3$$

$$\boxed{a = 7}$$

Given

$$\text{Sum of three terms} = \frac{91}{3}$$

$$\frac{a}{r} + a + ar = \frac{91}{3}$$

$$a\left(\frac{1}{r} + 1 + r\right) = \frac{91}{3}$$

$$\frac{1}{r} + 1 + r = \frac{91}{3a}$$

Put $a = 7$

$$\frac{1 + r + r^2}{r} = \frac{91}{3 \times 7}$$

$$\frac{1 + r + r^2}{r} = \frac{13}{3}$$

$$3 + 3r + 3r^2 = 13r$$

$$3r^2 - 13r + 3r + 3 = 0$$

$$3r^2 - 10r + 3 = 0$$

$$(r - 3)\left(r - \frac{1}{3}\right) = 0$$

9

$-9 \quad -1$

$\frac{-9}{3}, \quad \frac{-1}{3}$

$-3, \quad \frac{-1}{3}$

$r - 3 = 0 \qquad r - \frac{1}{3} = 0$

$\mathbf{r = 3} \qquad \mathbf{r = \frac{1}{3}}$

$\therefore$ **3 terms**

$a = 7, r = 3$	$a = 7, r = \frac{1}{3}$
$\frac{a}{r}, a, ar$	
$\frac{7}{3}, 7, 7 \times 3$	$\frac{7}{\frac{1}{3}}, 7, 7 \times \frac{1}{3}$
$\mathbf{\frac{7}{3}, 7, 21}$	$\mathbf{21, 7, \frac{7}{3}}$

Type IV: Prove the following

Q.No: 8, 12

8. If a, b, c are in A.P then show that $3^a, 3^b, 3^c$ are in G.P.

Given

$$a, b, c \text{ are in A.P.}$$

$$b = \frac{a + c}{2}$$

$$2b = a + c \qquad ...(1)$$

Let

$3^a, 3^b, 3^c$ are in G.P.

$$(3^b)^2 = 3^a \cdot 3^c$$

$$3^{2b} = 3^{a + c}$$

[a, b, c are in G.P then $b^2 = ac$]

We have $2b = a + c$

$$3^{a + c} = 3^{a + c}$$

Hence proved.

12. If a, b, c are three consecutive terms of an A.P. and x, y, z are three consecutive terms of a G.P. then prove that

$$x^{b - c} \times y^{c - a} \times z^{a - b} = 1$$

Given

- a, b, c are three consecutive terms of an A.P.

$$a = a$$
$$b = a + d$$
$$c = a + 2d$$

- x, y, z are three consecutive terms of a G.P.

$$x = a$$
$$y = ar$$
$$z = ar^2$$

LHS

$$x^{b-c} \times y^{c-a} \times z^{a-b} = 1$$
$$= (a)^{(a+d)-(a+2d)} \times (ar)^{a+2d-a} \times (ar^2)^{a-(a+d)}$$
$$= a^{-d} \times a^{2d} \times r^{2d} \times a^{-d} \times r^{-2d}$$
$$= a^{-d+2d-d} \times r^{2d-2d}$$
$$= a^0 \times r^0$$
$$= 1 \text{ RHS}$$

Type IV: G.P and S.I and based sums
Q.No: 10, 11 Example 2.45

10. A man joined a company as Assistant Manager. The Company give him a starting salary of Rs. 60,000 and agreed to increase his salary 5% annually. What will be his salary after 5 years.

Salary (P) = Rs. 60,000

increment = 5%

$$\text{S.I} = 60{,}000 \times \frac{5}{100}$$

$\therefore$ Amount $= P + S.I$

$$= 60{,}000 + 60{,}000 \times \frac{5}{100}$$
$$= 60{,}000\left(1 + \frac{5}{100}\right)^2$$

Principal for 2^{nd} year = Amount of 1^{st} year

$$\text{S.I}_2 = 60000\left(1 + \frac{5}{100}\right) \times \frac{5}{100}$$

Amount

$$= 60000\left(1 + \frac{5}{100}\right) + 60000\left(1 + \frac{5}{100}\right) \times \frac{5}{100}$$
$$= 60000\left(1 + \frac{5}{100}\right)\left[1 + \frac{5}{100}\right]$$
$$= 60000\left(1 + \frac{5}{100}\right)^2$$

Salary (principal) for 3^{rd} year

$$= 60000\left(1 + \frac{5}{100}\right)^2$$

Continuing this process we get a G.P. with

$$a = 60{,}000,\ r = \left(1 + \frac{5}{100}\right)$$

Salary after 5 years

$$t_6 = ar^5$$
$$= 60000\left(1 + \frac{5}{100}\right)^5$$
$$= 60{,}000 \times \left(\frac{21}{20}\right)^5$$
$$= 60{,}000 \times \frac{21}{20} \times \frac{21}{20} \times \frac{21}{20} \times \frac{21}{20} \times \frac{21}{20}$$
$$= 76{,}576.89$$
$$= \text{Rs. } 76{,}577$$

Salary after 5 years is Rs.76,577

11. Sivamani is attending an interview for a job and the company gave two offers to him.
Offer A: Rs. 20,000 to start with followed by a guaranteed annual increase of 6% for the first 5 years.
Offer B: Rs. 22,000 to start with followed by a guaranteed annual increase of 3% for the first 5 years.
What is his salary in the 4^{th} year with respect to the offers A and B?

Offer: A

Salary (P) = Rs. 20,000

$$S_n = a + a + a + \ldots + a$$

$$\boxed{S_n = na}$$

2. Sum to infinite terms of a G.P.

$$S_\infty = \frac{a}{1-r}, \quad -1 < r < 1$$

Type I: Find $S_n = \dfrac{a\,(r^n - 1)}{r-1}$ based sums

Q.No: (1) (i) (ii), (2), Example 2.46, Example 2.47, 3, 7, 8, Example 2.48, Example 2.52, 2.53

1. Find the sum of first n terms of the G.P.

(i) $5, -3, \frac{9}{5}, -\frac{27}{25}, \ldots$

$$a = 5$$

$$r = \frac{-3}{5}$$

$$S_n = \frac{a\,(r^n - 1)}{r-1}$$

$$= \frac{5\left(\left(\frac{-3}{5}\right)^n - 1\right)}{\frac{-3}{5} - 1}$$

$$= \frac{5\left(\left(\frac{-3}{5}\right)^n - 1\right)}{\frac{-8}{5}}$$

$$= 5 \times \frac{5}{-8}\left[\left(\frac{-3}{5}\right)^n - 1\right]$$

$$= \frac{25}{8}\left[1 - \left(\frac{-3}{5}\right)^n\right]$$

(ii) 256, 64, 16, …

$$a = 256$$

$$r = \frac{64}{256} = \frac{1}{4}$$

$$\boxed{S_n = \frac{a\,(r^n - 1)}{r-1}}$$

$$= \frac{256\left(\left(\frac{1}{4}\right)^n - 1\right)}{\frac{1}{4} - 1}$$

$$= \frac{256\left[\left(\frac{1}{4}\right)^n - 1\right]}{\frac{-3}{4}}$$

$$= 256 \times \frac{4}{-3}\left[\left(\frac{1}{4}\right)^n - 1\right]$$

$$= \frac{1024}{3}\left[1 - \left(\frac{1}{4}\right)^n\right]$$

2. Find the sum of first six terms of the G.P. 5, 15, 45, …

$$a = 5$$

$$r = \frac{15}{5} = 3$$

$$n = 6$$

$$S_n = \frac{a\,(r^n - 1)}{r-1}$$

$$= \frac{5\,(3^6 - 1)}{3-1}$$

$$= \frac{5 \times 728}{2}$$

$$= 5 \times 364$$

$$= 1820$$

Example 2.46

***Find the sum of 8 terms of the G.P.* 1, – 3, 9, – 27, …**

$$a = 1$$

$$r = -3$$

$$n = 8$$

$$\boxed{S_n = \frac{a\,(r^n - 1)}{r - 1}}$$

$$= \frac{1\,((-3)^8 - 1}{-3 - 1}$$

$$= \frac{6561 - 1}{-4}$$

$$= \frac{6560}{-4}$$

$$= -1640$$

Example 2.47

Find the first term of a G.P in which $S_6 = 4095$ and $r = 4$.

$$S_6 = 4095$$

$$r = 4$$

$$n = 6$$

$$a = ?$$

$$\boxed{S_n = \frac{a\,(r^n - 1)}{r - 1}}$$

$$4095 = \frac{a\,(4^6 - 1)}{4 - 1}$$

$$4095 \times 3 = a\,(4096 - 1)$$

$$4095 \times 3 = 4095a$$

$$\boxed{a = 3}$$

3. Find the first term of the G.P. whose common ratio 5 and whose sum to first 6 terms is 46872.

Given

$$r = 5$$

$$n = 6$$

$$S_n = 46872$$

$$a = ?$$

$$\boxed{S_n = \frac{a\,(r^n - 1)}{r - 1}}$$

$$46872 = \frac{a\,(5^6 - 1)}{5 - 1}$$

$$46872 = \frac{a\,(15625 - 1)}{4}$$

$$46872 = \frac{15624a}{4}$$

$$46872 = 3906a$$

$$\frac{46872}{3906} = a$$

$$\boxed{12 = a}$$

7. Find the sum of the Geometric series $3 + 6 + 12 + \ldots + 1536$.

$$a = 3$$

$$r = \frac{6}{3} = 2$$

$$t_n = 1536$$

$$ar^{n-1} = 1536$$

$$3\,(2)^{n-1} = 1536$$

$$2^{n-1} = \frac{1536}{3}$$

$$2^{n-1} = 512$$

$$2^{n-1} = 2^9$$

$$n - 1 = 9$$

$$n = 9 + 1$$

$$\boxed{n = 10}$$

$$\therefore S_n = \frac{a\,(r^n - 1)}{r - 1}$$

$$= \frac{3\,(2^{10} - 1)}{2 - 1}$$

$$= 3\,(1024 - 1)$$

$$= 3 \times 1023$$

$$S_{10} = 3069$$

8. Kumar writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they continue the process similarly. Assuming that the process is unaltered and it costs Rs. 2 to mail one letter, find the amount spent on postage when 8th set of letters is mailed.

Given data

$4, 4 \times 4, 4 \times 4 \times 4, \ldots$ 8 terms

$4, 4^2, 4^3, 4^4 \ldots$ is a G.P.

$$a = 4$$
$$r = 4$$
$$n = 8$$
$$S_n = \frac{a\,(r^n - 1)}{r - 1}$$
$$= 4\,\frac{(4^8 - 1)}{4 - 1}$$
$$= \frac{4}{3} \times 65535$$
$$= 4 \times 21845$$
$$= 87{,}380$$

Cost of 1 mail $=$ Rs. 2

$\therefore$ Total cost $= 87380 \times 2$

$=$ Rs. 1,74,760

Example 2.48

How many terms of the series* $1 + 4 + 16 + \ldots$ *make the sum 1365?

$$a = 1$$
$$r = 4$$
$$S_n = 1365$$
$$n = ?$$
$$S_n = \frac{a\,(r^n - 1)}{r - 1}$$
$$1365 = \frac{1\,(4^n - 1)}{4 - 1}$$
$$1365 = \frac{4^n - 1}{3}$$
$$(1365 \times 3) + 1 = 4^n$$
$$4095 + 1 = 4^n$$
$$4096 = 4^n$$
$$4^6 = 4^n$$
$$\boxed{n = 6}$$

4	4096
4	1024
4	256
4	64
4	16
	4

Example 2.52

***Find the least positive integer in such that* $1 + 6 + 6^2 + \ldots + 6^n > 5000$**

Here

$$S_n > 5000$$
$$\frac{a\,(r^n - 1)}{r - 1} > 5000$$
$$\frac{1\,(6^n - 1)}{6 - 1} > 5000$$
$$6^n - 1 > 5000 \times 5$$
$$6^n - 1 > 25000$$
$$6^n > 25001$$

we have $6^5 = 17776$; $6^6 = 46656$

$\therefore \quad 6^6 > 25001$

$\therefore$ Least value of n is 6.

Example 2.53

A person saved money every year, half as much as he could in the previous year. If he had totally saves Rs.7875 in 6 years then how much did he save in the first year.

Given

$$n = 6$$
$$S_6 = \text{Rs. } 7875$$

$$r = \frac{1}{2} \text{ then } a = ?$$

$$S_6 = 7875$$

$$\frac{a\,(r^n - 1)}{r - 1} = 7875$$

$$\frac{a\left(\left(\frac{1}{2}\right)^n - 1\right)}{\frac{1}{2} - 1} = 7875$$

$$\frac{a\left(\frac{1}{26} - 1\right)}{\frac{-1}{2}} = 7875$$

$$-2a\left(\frac{1}{64} - 1\right) = 7875$$

$$a\left(1 - \frac{1}{64}\right) = \frac{7875}{2}$$

$$a \times \frac{63}{64} = \frac{7875}{2}$$

$$a = \frac{7875}{2} \times \frac{64}{63}$$

$$a = \frac{252000}{63}$$

$$a = 4000$$

$\therefore$ The amount saved in the first year is Rs. 4000

Type II: $S_\infty = \dfrac{a}{1 - r}$ based sums

Q.No: 4 (i) (ii), Example 2.49, 5, 9, Example 2.50

4. Find the sum to infinity of

(i) $9 + 3 + 1 + \ldots$

$$a = 9$$

$$r = \frac{3}{9} = \frac{1}{3}$$

$$\boxed{S_\infty = \frac{a}{1 - r}}$$

$$= \frac{9}{1 - \frac{1}{3}}$$

$$= \frac{9}{\frac{2}{3}}$$

$$= 9 \times \frac{3}{2}$$

$$S_\infty = \frac{27}{2}$$

(ii) $21 + 14 + \frac{28}{3} + \ldots$

$$a = 21$$

$$r = \frac{14}{21} = \frac{2}{3}$$

$$\boxed{S_\infty = \frac{a}{1 - r}}$$

$$= \frac{21}{1 - \frac{2}{3}}$$

$$= \frac{21}{\frac{1}{3}}$$

$$= 21 \times 3$$

$$S_\infty = 63$$

Example 2.49

Find the sum $3 + 1 + \frac{1}{3} + \ldots \infty$

$$a = 3$$

$$r = \frac{1}{3}$$

$$\boxed{S_\infty = \frac{a}{1 - r}}$$

$$= \frac{3}{1 - \frac{1}{3}}$$

$$= \frac{3}{\frac{2}{3}}$$

$$= 3 \times \frac{3}{2}$$

$$= \frac{9}{2}$$

5. If the first term of an infinite G.P is 8 and its sum to infinity is $\frac{32}{3}$ then find the common ratio.

Given

$$a = 8$$

$$S_\infty = \frac{32}{3}$$

$$r = ?$$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

$$\frac{32}{3} = \frac{8}{1-r}$$

$$\frac{4}{3} = \frac{1}{1-r}$$

$$4(1-r) = 3$$

$$4 - 4r = 3$$

$$-4r = 3 - 4$$

$$-4r = -1$$

$$\boxed{r = \frac{1}{4}}$$

9. Find the rational form of the number $0 \cdot \overline{123}$.

We can write as $0 \cdot \overline{123}$ as

$$0.123\ 123\ 123 = 0.123 + 0.000123 + 0.000\ 000\ 123 + \dots \infty$$

$$a = 0.123$$

$$r = \frac{0.000123}{0.123}$$

$$= \frac{0.123}{123}$$

$$= 0.001$$

$$\boxed{S_\infty = \frac{a}{1-r}}$$

$$= \frac{0.123}{1-0.001}$$

$$= \frac{0.123}{0.999}$$

$$= \frac{123}{999}$$

$$0 \cdot \overline{123} = \frac{41}{333}$$

Example 2.50

***Find the rational forms of the number* 0.6666 …**

We can write 0.666 … as

$0.666 \ldots = 0.6 + 0.06 + 0.006 + \ldots + \infty$ (It is a G.P)

$$a = 0.6$$

$$r = \frac{0.06}{0.6}$$

$$= 0.1$$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{0.6}{1-0.1}$$

$$= \frac{0.6}{0.9}$$

$$= \frac{6}{9}$$

$$0.666 \ldots = \frac{2}{3}$$

Type III: Find the sum to *n* terms

Example 2.51, (6) (ii) (i), 10

Example 2.51

***Find the sum to n terms of the series* $5 + 55 + 555 + \ldots$**

$$= 5 + 55 + 555 + \ldots n \text{ terms}$$

$$= 5(1 + 11 + 111 + \ldots n \text{ terms})$$

$= 5 \times \frac{9}{9} (1 + 11 + 111 + \dots n \text{ terms})$

$= \frac{5}{9} [9 + 99 + 999 + \dots n \text{ terms}]$

$= \frac{5}{9} [(10 - 1) + (100 - 1) + (1000 - 1) + \dots n]$

$= \frac{5}{9} [(10 + 100 + 1000 + \dots n) - (1 + 1 + 1 + \dots + n)]$

It is a G.P

$a = 10$

$r = \frac{100}{10}$

$r = 10$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{5}{9}\left[\frac{10(10^n - 1)}{10 - 1}\right]$

$= \frac{5}{9}\left[\frac{10(10^n - 1)}{9} - n\right]$

$= \frac{50}{81}(10^n - 1) - \frac{5n}{9}$

6. Find the sum to n terms of the series

(ii) $3 + 33 + 333 + \dots$ to n terms

$= 3 + 33 + 333 + \dots n \text{ terms}$

$= 3 [1 + 11 + 111 + \dots n]$

$= \frac{3}{9} \times 9 [1 + 11 + 111 + \dots n]$

$= \frac{3}{9} [9 + 99 + 999 + \dots n \text{ terms}]$

$= \frac{1}{3} [(10 - 1) + (100 - 1) + (1000 - 1) + \dots n]$

$= \frac{1}{3} [(10 + 100 + 1000 + \dots n) - (1 + 1 + 1 + \dots n)]$

$a = 10$

$r = \frac{100}{10}$

$r = 10$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1}{3}\left[\frac{10(10^n - 1)}{10 - 1} - n\right]$

$= \frac{1}{3}\left[\frac{10(10^n - 1)}{9} - n\right]$

$= \frac{10(10^n - 1)}{27} - \frac{n}{3}$

(i) $0.4 + 0.44 + 0.444 + \dots$ to n terms

$= 0.4 + 0.44 + 0.444 + \dots \text{ to } n \text{ terms}$

$= 4 (0.01 + 0.11 + 0.111 + \dots n \text{ terms})$

$= \frac{4}{9} \times 9 (0.1 + 0.11 + 0.111 + \dots n \text{ terms})$

$= \frac{4}{9} [0.9 + 0.99 + 0.999 + \dots n \text{ terms}]$

$= \frac{4}{9} [(1 - 0.1) + (1 - 0.01) + (1 - 0.001) + \dots n]$

$= \frac{4}{9} [(1 + 1 + \dots n) - (0.1 + 0.01 + 0.001 + \dots n)]$

$= \frac{4}{9}\left[n - 0.1\left(\frac{1 - 0.1^n}{1 - 0.1}\right)\right]$

$= \frac{4}{9}\left[n - 0.1\left(\frac{1 - \frac{1}{10^n}}{0.9}\right)\right]$

$= \frac{4}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right]$

$= \frac{4n}{9} - \frac{4}{81}\left(1 - \frac{1}{10^n}\right)$

It is a G.P

$a = 0.1$

$r = 0.1$

$S_n = \frac{a(1 - r^n)}{1 - r}$

10. If $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2 y + xy^2 + y^3) + \dots n$ terms then prove that

$$(x - y) S_n = \left[\frac{x^2(x^n - 1)}{x - 1} - \frac{y^2(y^n - 1)}{y - 1}\right]$$

Given

$S_n = (x + y) + (x^2 + xy + y^2)$

$+ (x^3 + x^2 y + xy^2 + y^3) + \dots + n$

- $xS_n = x(x + y) + x(x^2 + xy + y^2)$

$+ x(x^3 + x^2 y + xy^2 + y^3) + \dots$

$= x^2 + xy + x^3 + x^2 y + xy^2 + x^4 + x^3 y$

$+ x^2 y^2 + xy^3 + \dots$

- $yS_n = y(x+y) + y(x^2 + xy + y^2)$
$$+ y(x^3 + x^2 y + xy^2 + y^3) + \ldots$$
$$= xy + y^2 + x^2 y + xy^2 + y^3 + x^3 y + x^2 y^2 + xy^3 + y^4 + \ldots$$

Now

$$xS_n - yS_n = (x^2 + x^3 + x^4 + \ldots) - (y^2 + y^3 + y^4 + \ldots)$$

$$(x-y)\,S_n = \left[\frac{x^2(x^n - 1)}{x-y} - \frac{y^2(y^n-1)}{y-1}\right] \text{proved}$$

Note

- $x^2 + x^3 + x^4 + \ldots$ is a G.P. with $a = x^2, r = x$
- $y^2 + y^3 + y^4 + \ldots$ is a G.P. with $a = y^2, r = y$

use $S_n = \dfrac{a(r^n - 1)}{r-1}$

Exercise 2.9

KEY POINTS

Special series

1. Sum of first 'n' natural numbers
$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
2. Sum of first 'n' odd natural numbers
$$1 + 3 + 5 + \ldots + (2n-1) = n^2$$
3. Sum of squares of first 'n' natural numbers
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
4. Sum of cubes of first 'n' natural numbers
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Note:

(i) $\Sigma n^3 = (\Sigma n)^2$

(ii) $\Sigma n = \sqrt{\Sigma n^3}$

Type I: Problems based on sum of first *n* natural numbers

Q.No: (1) (i) (ii) (iii), Example 2.54 (i) (ii) Example 2.55 (ii)

1. Find the sum of the following series:

(i) $1 + 2 + 3 + \ldots + 60$

$$n = 60$$
$$\boxed{\Sigma n = \frac{n(n+1)}{2}}$$
$$= \frac{60 \times 61}{2}$$
$$= 30 \times 61$$
$$= 1830$$

(ii) $3 + 6 + 9 + \ldots + 96$

$$3 + 6 + 9 + \ldots + 96$$
$$= 3\,[1 + 2 + 3 + \ldots + 32]$$
$$n = 32$$
$$\boxed{\Sigma n = \frac{n(n+1)}{2}}$$
$$= 3 \times \frac{32 \times 33}{2}$$
$$= 3 \times 16 \times 33$$
$$= 1584$$

(iii) $51 + 52 + 53 + \ldots + 92$

$$= 51 + 52 + 53 + \ldots + 92$$
$$= (1 + 2 + 3 + \ldots + 92) - (1 + 2 + 3 + \ldots + 50)$$
$$n = 92\,;\; n = 50$$
$$\Sigma n = \frac{n(n+1)}{2}$$
$$= \left(\frac{92 \times 93}{2}\right) - \left(\frac{50 \times 51}{2}\right)$$
$$= (46 \times 93) - (25 \times 51)$$
$$= 4278 - 1275$$
$$= 3003$$

Example 2.54

Find the value of (i) $\mathbf{1 + 2 + 3 + \ldots + 50}$

$$n = 50$$

$$\Sigma n = \frac{n(n+1)}{2}$$

$$= \frac{50 \times 51}{2}$$

$$= 25 \times 51$$

$$= 1275$$

(ii) $\mathbf{16 + 17 + 18 + \ldots + 75}$

$$= (1 + 2 + 3 + \ldots + 75) - (1 + 2 + 3 + \ldots + 15)$$

$n = 75$; $n = 15$

$$\Sigma n = \frac{n(n+1)}{2}$$

$$= \left(\frac{75 \times 76}{2}\right) - \left(\frac{15 \times 16}{2}\right)$$

$$= (75 \times 38) - (15 \times 8)$$

$$= 2850 - 120$$

$$= 2730$$

Example 2.55

Find the sum of

(ii) $\mathbf{2 + 4 + 6 + \ldots + 80}$

$$2 + 4 + 6 + \ldots + 80 = 2(1 + 2 + 3 + \ldots + 40)$$

$$n = 40$$

$$\Sigma n = \frac{n(n+1)}{2}$$

$$= 2 \times \frac{40 \times 41}{2}$$

$$= 1640$$

Type II: Sum of first 'n' odd natural numbers

Q.No: (1), (vii), Example 2.55 (i) (iii)

1. Find the sum of the following series

(vii) $\mathbf{1 + 3 + 5 + \ldots + 71}$

It is an A.P with $a = 1, d = 3 - 1 = 2, l = 71$

$$n = \left(\frac{l - a}{d}\right) + 1$$

$$= \left(\frac{71 - 1}{2}\right) + 1$$

$$= \frac{70}{2} + 1$$

$$= 35 + 1$$

$$n = 36$$

$$\therefore S_n = n^2$$

$$= (36)^2$$

$$= 1296$$

Example 2.55

(i) Find the sum of $\mathbf{1 + 3 + 5 + \ldots}$ *to 40 terms*

$$1 + 3 + 5 + \ldots \text{ 40 terms}$$

$$S_n = n^2$$

$$= (40)^2$$

$$= 1600$$

(iii) $\mathbf{1 + 3 + 5 + \ldots + 55}$

It is an A.P with $a = 1$, $d = 3 - 1 = 2$; $l = 55$

$$n = \left(\frac{l - a}{d}\right) + 1$$

$$= \left(\frac{55 - 1}{2}\right) + 1$$

$$= \left(\frac{54}{2}\right) + 1$$

$$= 27 + 1$$

$n = 28$

$S_n = n^2$

$= (28)^2$

$= 784$

Type III: Sum of squares of first '*n*' natural numbers

(1) (iv) (v), Example 2.56 (i) (ii) (iii), 6

1. Find the sum of the following series

(iv) $1 + 4 + 9 + 16 + \ldots + 225$

$$1 + 4 + 9 + 16 + \ldots + 225$$
$$= 1^2 + 2^2 + 3^2 + 4^2 + \ldots + 15^2$$
$$n = 15$$
$$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{15 \times 16 \times 31}{6}$$
$$= 1240$$

(v) $6^2 + 7^2 + 8^2 + \ldots + 21^2$

$$= (1^2 + 2^2 + 3^2 + \ldots + 21^2) - (1^2 + 2^2 + \ldots + 5^2)$$
$$n = 21 \;;\; n = 5$$
$$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$= \left(\frac{21 \times 22 \times 43}{6}\right) - \left(\frac{5 \times 6 \times 11}{6}\right)$$
$$= 3311 - 55$$
$$= 3256$$

Example 2.56

Find the sum of

***(i)* $1^2 + 2^2 + \ldots + 19^2$**

$$n = 9$$
$$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{19 \times 20 \times 39}{6}$$
$$= 19 \times 10 \times 13$$
$$= 2470$$

(ii) $5^2 + 10^2 + 15^2 + \ldots + 105^2$

$$5^2 + 10^2 + 15^2 + \ldots + 105^2$$
$$= 5^2\,(1^2 + 2^2 + 3^2 + \ldots + 21^2)$$
$$n = 21$$
$$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$= 25 \times \frac{21 \times 22 \times 43}{6}$$
$$= 25 \times 7 \times 11 \times 43$$
$$= 82775$$

(iii) $15^2 + 16^2 + 17^2 + \ldots + 28^2$

$$= (1^2 + 2^2 + 3^2 + \ldots + 28^2) - (1^2 + 2^2 + \ldots + 14^2)$$
$$n = 28 \;;\; n = 14$$
$$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$= \left(\frac{28 \times 29 \times 57}{6}\right) - \left(\frac{14 \times 15 \times 29}{6}\right)$$
$$= (14 \times 29 \times 19) - (7 \times 5 \times 29)$$
$$= 7714 - 1015$$
$$= 6699$$

6. Rekha has 15 square colour papers of sizes 10 cm, 11 cm, 12 cm, … 24 cm. How much area can be decorated with these colour papers?

Area of square $= a^2$ sq.u

Total area $= 10^2 + 11^2 + 12^2 + \ldots + 24^2$

$$= (1^2 + 2^2 + 3^2 + \ldots + 24^2) - (1^2 + 2^2 + \ldots + 9^2)$$
$$n = 24 \;;\; n = 9$$

$$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \left(\frac{24 \times 25 \times 49}{6}\right) - \left(\frac{9 \times 10 \times 19}{6}\right)$$

$$= (4 \times 25 \times 49) - (3 \times 5 \times 19)$$

$$= 4900 - 285$$

$$= 4615$$

$\therefore$ Area can be deconated is 4615 cm^2

Type IV: Sum of cubes of first *n* natural numbers

Q.No: (1) (vi), Example 2.57 (i) (ii)

1. Find the sum of the following series

(vi) $10^3 + 11^3 + 12^3 + \ldots + 20^3$

$$10^3 + 11^3 + 12^3 + \ldots + 20^3$$

$$= (1^3 + 2^3 + \ldots + 20^3) - (1^3 + 2^3 + \ldots + 9^3)$$

$n = 20, \; n = 9$

$$\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= \left(\frac{20 \times 21}{2}\right)^2 - \left(\frac{9 \times 10}{2}\right)^2$$

$$= (210)^2 - (45)^2$$

$$= 44100 - 2025$$

$$= 42,075$$

Example 2.57

Find the sum of

(i) **$1^3 + 2^3 + 3^3 + \ldots + 16^3$**

$n = 16$

$$\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= \left(\frac{16 \times 7}{2}\right)^2$$

$$= (8 \times 17)^2$$

$$= (136)^2$$

$$= 18496$$

(ii) $9^3 + 10^3 + \ldots + 21^3$

$$= (1^3 + 2^3 + \ldots + 21^3) - (1^3 + 2^3 + \ldots + 8^3)$$

$n = 21 \; ; \; n = 8$

$$\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= \left[\frac{21 \times 22}{2}\right]^2 - \left[\frac{8 \times 9}{2}\right]^2$$

$$= (21 \times 11)^2 - (4 \times 9)^2$$

$$= (231)^2 - (36)^2$$

$$= 53,361 - 1296$$

$$= 52,065$$

Type V: Combined sums $\Sigma n, \Sigma n^2, \Sigma n^3$

Q.No: 2, 3, Example 2.58, 4, 5 and 7.

2. If $1 + 2 + 3 + \ldots + k = 325$, then find $1^3 + 2^3 + 3^3 + \ldots + k^3$.

Given

$$1 + 2 + 3 + \ldots + k = 325$$

$$\frac{k(k+1)}{2} = 325$$

Squaring on both sides

$$\left[\frac{k(k+1)}{2}\right]^2 = (325)^2$$

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = 1,05,625$$

3. If $1^3 + 2^3 + 3^3 + \ldots + k^3 = 44100$ then find $1 + 2 + 3 + \ldots + k$.

Given

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = 44100$$

$$\left[\frac{k(k+1)}{2}\right]^2 = 44100$$

$$\frac{k(k+1)}{2} = \sqrt{44100}$$

$$1 + 2 + 3 + \ldots + k = 210$$

Example 2.58

If $1 + 2 + 3 + \ldots + n = 666$ then find 'n'

Given

$$1 + 2 + 3 + \ldots + n = 666$$

$$\frac{n(n+1)}{2} = 666$$

$$n(n+1) = 666 \times 2$$

$$n^2 + n = 1332$$

$$n^2 + n - 1332 = 0$$

$$(n+37)(n-36) = 0$$

$n + 37 = 0$	$n - 36 = 0$
$n = -37$	$\mathbf{n = 36}$
not possible	

4. How many terms of the series $1^3 + 2^3 + 3^3 + \ldots$ should be taken to get the sum 14400?

Given

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = 14400$$

$$\left[\frac{n(n+1)}{2}\right]^2 = 14400$$

$$\frac{n(n+1)}{2} = \sqrt{14400}$$

$$\frac{n(n+1)}{2} = 120$$

$$n^2 + n = 240$$

$$n^2 + n - 240 = 0$$

$$(n+16)(n-15) = 0$$

$n + 16 = 0$	$\mathbf{n - 15 = 0}$
$n = -16$	$\mathbf{n = 15}$
not possible	

5. The sum of the squares of the first n natural numbers is 285, while the sum of their cubes is 2025. Find the values of n.

Given

$$\Sigma n^2 = 285 \Rightarrow \frac{n(n+1)(2n+1)}{6} = 285 \quad \ldots(1)$$

$$\Sigma n^3 = 2025 \Rightarrow \left[\frac{n(n+1)}{2}\right]^2 = 2025$$

$$\frac{n(n+1)}{2} = \sqrt{2025}$$

$$\frac{n(n+1)}{2} = 45$$

$$n(n+1) = 90 \quad \ldots(2)$$

Put (2) in (1)

$$\frac{90(2n+1)}{6} = 285$$

$$15(2n+1) = 285$$

$$2n + 1 = \frac{285}{15}$$

$$2n + 1 = 19$$

$$2n = 19 - 1$$

$$2n = 18$$

$$n = \frac{18}{2}$$

$$\boxed{n = 9}$$

7. Find the sum of the series $(2^3 - 1) + (4^3 - 3^3) + (6^3 - 5^3) + \ldots$ to (i) n terms (ii) 8 terms

$$= (2^3 - 1) + (4^3 - 3^3) + (6^3 - 5^3) + n$$

$$= \sum_{1}^{n} (2^3 + 4^3 + 6^3 + \dots)$$

$$- \sum_{1}^{n} (1^3 + 3^3 + 5^3 + \dots)$$

$$= \sum_{1}^{n} [(2n)^3 - (2n-1)^3]$$

use $\boldsymbol{a^3 - b^3 = (a-b)^3 + 3ab\,(a-b)}$

$= (2n - 2n + 1)^3 + 3\,(2n)\,(2n-1)\,(2n - 2n + 1)$

$= 1 + 3\,(2n)\,(2n-1)\,(1)$

$= 1 + 6n\,(2n-1)$

$= 1 + 12n^2 - 6n$

$$\therefore \sum_{1}^{n} (1 + 12n^2 - 6n)$$

$$= n + 12\left[\frac{n\,(n+1)\,(2n+1)}{6}\right] - 6\left[\frac{n\,(n+1)}{2}\right]$$

$= n + 2\,[n\,(n+1)\,(2n+1)] - 3n\,(n+1)$

$= n + 2\,[(n^2 + n)\,(2n+1)] - 3n^2 - 3n$

$= n + 2\,[2n^3 + n^2 + 2n^2 + n] - 3n^2 - 3n$

$= n + 4n^3 + 2n^2 + 4n^2 + 2n - 3n^2 - 3n$

$= 4n^3 + 3n^2$

Sum of first 'n' terms

When $n = 8$

Sum $= 4\,(8)^3 + 3\,(8)^3$

$= 4\,(512) + 3\,(64)$

$= 2048 + 192$

$= 2240$

Exercise 2.10

Multiple choice questions

1. Euclid's division lemma states that for positive integers a and b, there exist unique integers q and r such that $n = bq + r$ here r must satisfy.

1. $1 < r < b$
2. $0 < r < b$
3. $0 \le r < b$
4. $0 < r \le b$

Solution:

1. Condition: $0 \le r < b$ **Ans: (2) $0 \le r < b$**

2. Using Euclid's division lemma, if the cube of any positive integer is divided by 9 then the possible remainders are

1. 0, 1, 8
2. 1, 4, 8
3. 0, 1, 3
4. 1, 3, 5

✍ **Solution:**

Cube of any + ve integers

$1^3, 2^3, 3^3, 4^3, \dots$

1, 8, 27, 64, 125, 216, …

Divided by 9, the possible

remainders: $\frac{27}{9} \Rightarrow 0$; $\frac{64}{9} \Rightarrow 1 = \frac{125}{9} \Rightarrow 8$

Ans: (1) 0, 1, 8

3. If the HCF of 65 and 117 is expressible in the form of $65m - 117$, then the value of m is

1. 4
2. 2
3. 1
4. 3

✍ **Solution:**

HCF of 65 and 117

$117 = 65 \times 1 + 52$

$65 = 52 \times 1 + 13$

$52 = 13 \times 4 + 0$

$\therefore$ HCF is 13

CHAPTER 3

ALGEBRA

Exercise 3.1

Key points

1. Linear Equation in two variables

Any first degree equation containing two variables x and y is called a "linear equation" in two variables. The general form of linear equation in two variables x and y is $\boldsymbol{ax + by + c = 0}$, where atleast one of a, b, c is non-zero and a, b, c are real numbers.

Note:

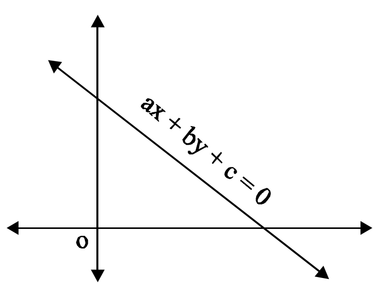
- $xy - 7 = 3$ is not a linear equation in two variables since the term xy is of degree 2.
- A linear equation in two variables represent a straight line in xy plane.

2. System of linear equations in three variables

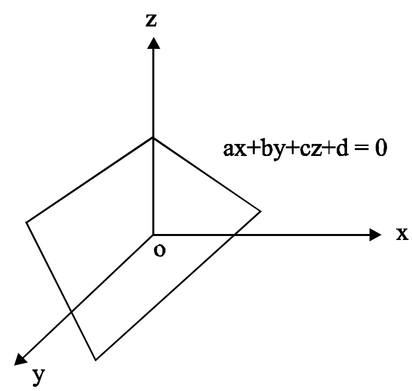
The general form of a linear equation in three variables x, y and z is $ax + by + cz + d = 0$ where a, b, c, d are real numbers, and atleast one of a, b, c is non-zero.

Note:

- A linear equation in two variables of the form $ax + by + c = 0$ represents a straight line.



- A linear equation in three variables of the form $ax + by + cz + d = 0$, represents a plane.



Note:

- If you obtain a false equation such as $0 = 1$, in any of the step then the system has **no solution.**
- If you do not obtain a false solution, but obtain an identity, such as $0 = 0$ then the system has **infinitely many** solution.

Type I: Solve the linear equations in two variables

Example 3.1, 3.2

Example 3.2

Solve $\boldsymbol{2x - 3y = 6 \,;\, x + y = 1}$

$$2x - 3y = 6 \quad \text{...(1)}$$

$$x + y = 1 \quad \text{...(2)}$$

$$(1) \Rightarrow 2x - 3y = 6$$

$$(2) \times 3 \Rightarrow 3x + 3y = 3$$

$$5x = 9$$

$$\boxed{x = \frac{9}{5}}$$

$$(1) \Rightarrow 2x - 3y = 6$$

$$(2) \times 2 \Rightarrow 2x + 2y = 2$$

$$-5y = 4$$

$$\boxed{y = \frac{-4}{5}}$$

Example 3.1

The father's age is six times his son's age six years hence the age of father will be four times his son's age. Find the present ages (in years) of the son and father.

Let present age of father = 'x' years

present age of son = 'y' years

Given

$$x = 6y \quad \text{...(1)}$$

$$x + 6 = 4(y + 6) \quad \text{... (2)}$$

Put (1) in (2)

$$6y + 6 = 4y + 24$$

$$6y - 4y = 24 - 6$$

$$2y = 18$$

$$\boxed{y = 9}$$

$$(1) \Rightarrow x = 6(9)$$

$$\boxed{x = 54}$$

$\therefore$ Son's age = 9 years

Father's age = 54 years

Type II: Solve the following system of linear equations in three variables.

Q.No: 1 (i) (ii) (iii), Example 3.3, 3.7, 3.8

1. (i) $\mathbf{x + y + z = 5 \,;\, 2x - y + z = 9 \,;\, x - 2y + 3z = 16}$

$$x + y + z = 5 \quad \text{...(1)}$$

$$2x - y + z = 9 \quad \text{...(2)}$$

$$x - 2y + 3z = 16 \quad \text{...(3)}$$

From (1) and (2) eliminate 'z'

$$x + y + z = 5$$
$$(-) \ (+) \ (-) \ (-)$$
$$2x - y + z = 9$$
$$\overline{-x + 2y = -4} \quad \text{...(4)}$$

From (2) & (3) eliminate 'z'

$$(2) \times 3 \Rightarrow 6x - 3y + 3z = 27$$
$$(-) \ (+) \ (-) \ (-)$$
$$(3) \Rightarrow x - 2y + 3z = 16$$
$$\overline{5x - y = 11} \quad \text{...(5)}$$

Solve (4) and (5)

$$(4) \Rightarrow -x + 2y = -4$$
$$(5) \times 2 \Rightarrow 10x - 2y = 22$$
$$\overline{9x = 18}$$

$$x = \frac{18}{9}$$

$$\boxed{x = 2}$$

$$(5) \Rightarrow 5(2) - y = 11$$

$$10 - y = 11$$

$$10 - 11 = y$$

$$\boxed{y = -1}$$

$$(1) \Rightarrow 2 - 1 + z = 5$$

$$1 + z = 5$$

$$z = 5 - 1$$

$$\boxed{z = 4}$$

$\therefore \quad x = 2 \,;\, y = -1;\, z = 4$

Example 3.3

Solve the following system of linear equations in three variables $\mathbf{3x - 2y + z = 2 \,;\, 2x + 3y - z = 5 \,;\, x + y + z = 6}$

$$3x - 2y + z = 2 \quad \text{...(1)}$$

$$2x + 3y - z = 5 \quad \text{...(2)}$$

$$x + y + z = 6 \quad \text{...(3)}$$

From (1) and (2) eliminate 'z'

$$3x - 2y + z = 2$$
$$2x + 3y - z = 5$$
$$\overline{5x + y = 7} \quad \text{...(4)}$$

From (2) and (3) eliminate 'z'

$$2x + 3y - z = 5$$
$$x + y + z = 6$$
$$3x + 4y = 11 \quad ...(5)$$

Solve (4) and (5)

$$(4) \times 4 \Rightarrow 20x + 4y = 28$$
$$(-) \quad (-) \quad (-)$$
$$(5) \Rightarrow 3x + 4y = 11$$
$$17x = 17$$
$$\boxed{x = 1}$$

$$(4) \Rightarrow 5\,(1) + y = 7$$
$$y = 7 - 5$$
$$\boxed{y = 2}$$

$$(3) \Rightarrow 1 + 2 + z = 6$$
$$3 + z = 6$$
$$z = 6 - 3$$
$$\boxed{z = 3}$$

$\therefore \quad x = 1\,;\, y = 2\,;\, z = 3$

1. (iii) $\boldsymbol{x + 20 = \frac{3y}{2} + 10 = 2z + 5 = 110 - (y + z)}$

Let

- $x + 20 = \frac{3y}{2} + 10$

$$2x + 40 = 3y + 20$$
$$2x - 3y = 20 - 40$$
$$2x - 3y = -20 \quad ...(1)$$

- $\frac{3y}{2} + 10 = 2z + 5$

$$3y + 20 = 4z + 10$$
$$3y - 4z = 10 - 20$$
$$3y - 4z = -10 \quad ...(2)$$

- $2z + 5 = 110 - (y + z)$

$$2z + 5 = 110 - y - z$$
$$y + z + 2z = 110 - 5$$
$$y + 3z = 105 \quad ...(3)$$

Solve (2) and (3)

$$(2) \Rightarrow 3y - 4z = -10$$
$$(-) \quad (-) \quad (-)$$
$$(3) \times 3 \Rightarrow 3y + 9z = 315$$
$$-13z = -325$$
$$z = \frac{325}{13}$$
$$\boxed{z = 25}$$

$$(3) \Rightarrow y + 3\,(25) = 105$$
$$y + 75 = 105$$
$$y = 105 - 75$$
$$\boxed{y = 30}$$

$$(1) \Rightarrow 2x - 3\,(30) = -20$$
$$2x - 90 = -20$$
$$2x = -20 + 90$$
$$x = \frac{70}{2}$$
$$\boxed{x = 35}$$

✍ **Solution:**

$x = 35\,;\, y = 30\,;\, z = 25$

1. (ii) $\boldsymbol{\frac{1}{x} - \frac{2}{y} + 4 = 0\,;\, \frac{1}{y} - \frac{1}{z} + 1 = 0\,;\, \frac{2}{z} + \frac{3}{x} = 14}$

Let $\frac{1}{x} = a\,;\, \frac{1}{y} = b\,;\, \frac{1}{z} = c$

$$a - 2b = -4 \quad ...(1)$$
$$b - c = -1 \quad ...(2)$$
$$3a + 2c = 14 \quad ...(3)$$

From (1) and (2) eliminate 'b'

$$(1) \Rightarrow a - 2b = -4$$
$$(2) \times 2 \Rightarrow 2b - 2c = -2$$
$$a - 2c = -6 \quad ...(4)$$

Solve (3) and (4)

$$3a + 2c = 14$$
$$a - 2c = -6$$
$$4a = 8$$
$$a = \frac{8}{4}$$
$$\boxed{a = 2}$$

$(1) \Rightarrow 2 - 2b = -4$

$$-2b = -4 - 2$$
$$b = \frac{-6}{-2}$$
$$\boxed{b = 3}$$

$(2) \Rightarrow 3 - c = -1$

$$3 + 1 = c$$
$$\boxed{4 = c}$$

Solution:

$a = 2$	$x = \frac{1}{2}$
$b = 3$	$y = \frac{1}{3}$
$c = 4$	$z = \frac{1}{4}$

Example 3.7

Solve $\frac{x}{2} - 1 = \frac{y}{6} + 1 = \frac{z}{7} + 2\,;\ \frac{y}{3} + \frac{z}{2} = 13$

Let $\frac{x}{2} - 1 = \frac{y}{6} + 1$

$$\frac{x-2}{2} = \frac{y+6}{6}$$
$$3x - 6 = y + 6$$
$$3x - y = 6 + 6$$
$$3x - y = 12 \qquad ...(1)$$

Let $\frac{y}{6} + 1 = \frac{z}{7} + 2$

$$\frac{y+6}{6} = \frac{z+14}{7}$$
$$7y + 42 = 6z + 84$$
$$7y - 6z = 84 - 42$$
$$7y - 6z = 42 \qquad ...(2)$$

Given

$$\frac{y}{3} + \frac{z}{2} = 13$$
$$\frac{2y + 3z}{6} = 13$$
$$2y + 3z = 78 \qquad ...(3)$$

Solve (2) and (3)

$$(2) \Rightarrow 7y - 6z = 42$$
$$(3) \times 2 \Rightarrow 4y + 6z = 156$$
$$11y = 198$$
$$y = \frac{198}{11}$$
$$\boxed{y = 18}$$

$$(1) \Rightarrow 3x - 18 = 12$$
$$3x = 12 + 18$$
$$3x = 30$$
$$\boxed{x = 10}$$

$(3) \Rightarrow 2\,(18) + 3z = 78$

$$36 + 3z = 78$$
$$3z = 78 - 36$$
$$3z = 42$$
$$z = \frac{42}{3}$$
$$\boxed{z = 14}$$

Solution:

$x = 10 \quad ; \quad y = 18 \quad ; \quad z = 14$

Example 3.8

Solve

$$\frac{1}{2x}+\frac{1}{4y}-\frac{1}{3z}=\frac{1}{4}\ ;\ \frac{1}{x}=\frac{1}{3y}\ ;\ \frac{1}{x}-\frac{1}{5y}+\frac{4}{z}=2\frac{2}{15}$$

Let $\frac{1}{x}=a\ ;\ \frac{1}{y}=b\ ;\ \frac{1}{z}=c$

$$\frac{a}{2}+\frac{b}{4}-\frac{c}{3}=\frac{1}{4}$$

Multiply by 12

$$6a+3b-4c=3 \quad \text{...(1)}$$

$$a=\frac{b}{3}$$

$$3a-b=0 \quad \text{...(2)}$$

$$a-\frac{b}{5}+4c=\frac{32}{15}$$

Multiply by 15

$$15a-3b+60c=32 \quad \text{...(3)}$$

From (1) and (3) eliminate 'c'

$(1)\times 15 \Rightarrow 90a+45b-60c=45$

$(3) \Rightarrow 15a-3b+60c=32$

$$105a+42b=77 \quad \text{...(4)}$$

$(2)\times 42 \Rightarrow 126a-42b=0$

$(4) \Rightarrow 105a+42b=77$

$$231a=77$$

$$a=\frac{77}{231}$$

$$\boxed{a=\frac{1}{3}}$$

$$(2) \Rightarrow 3\left(\frac{1}{3}\right)-b=0$$

$$1-b=0$$

$$\boxed{b=1}$$

$$6\left(\frac{1}{3}\right)+3\,(1)-4c=3$$

$$2+3-4c=3$$

$$5-4c=3$$

$$-4c=3-5$$

$$-4c=-2$$

$$c=\frac{2}{4}$$

$$\boxed{c=\frac{1}{2}}$$

$\therefore$ Solution

$a=\frac{1}{3}$	$x=3$
$b=1$	$y=1$
$z=\frac{1}{2}$	$z=2$

Type III: Discuss the nature of solutions

Q.No. 2(i) (ii) (iii), Example 3.5, 3.6

2. Discuss the nature of solutions of the following system of equations

(i) $\boldsymbol{x+2y-z=6\ ;\ -3x-2y+5z=-12\ ;\ x-2z=3}$

$$x+2y-z=6 \quad \text{...(1)}$$

$$-3x-2y+5z=-12 \quad \text{...(2)}$$

$$x-2z=3 \quad \text{...(3)}$$

From (1) and (2) eliminate 'y'

$$x+2y-z=6$$

$$-3x-2y+5z=-12$$

$$-2x+4z=-6$$

divide by (-2)

$$x-2z=3 \quad \text{...(4)}$$

Now solve (3) and (4)

$$x-2z=3$$

$$(-)\quad(+)\quad(-)$$

$$x-2z=3$$

$$0=0$$

Here we arrive at an identity 0 = 0. Hence the system has an infinite number of solutions.

Example 3.5

Solve $\boldsymbol{x + 2y - z = 5 \,;\, x - y + z = -2 \,;\, -5x - 4y + z = -11}$

$$x + 2y - z = 5 \quad ...(1)$$

$$x - y + z = -2 \quad ...(2)$$

$$-5x - 4y + z = -11 \quad ...(3)$$

From (1) and (2) eliminate 'z'

$$\begin{array}{r} x + 2y - z = 5 \\ x - y + z = -2 \\ \hline 2x + y = 3 \\ \hline \end{array} \quad ...(4)$$

From (2) and (3) eliminate 'z'

$$\begin{array}{r} x - y + z = -2 \\ (+) \quad (+) \quad (-) \quad (+) \\ -5x - 4y + z = -11 \\ \hline 6x + 3y = 9 \\ \hline \end{array}$$

divide by 3

$$2x + y = 3 \quad ...(5)$$

Solve (4) and (5)

$$\begin{array}{r} 2x + y = 3 \\ (-) \quad (-) \quad (-) \\ 2x + y = 3 \\ \hline 0 = 0 \\ \hline \end{array}$$

Here we arrive at an identity 0 = 0.

Hence the system has an infinite number of solutions.

Example 3.6

Solve $\boldsymbol{3x + y - 3z = 1, -2x - y + 2z = 1 \,;\, -x - y + z = 2}$

$$3x + y - 3z = 1 \quad ...(1)$$

$$-2x - y + 2z = 1 \quad ...(2)$$

$$-x - y + z = 2 \quad ...(3)$$

From (1) & (2) eliminate 'y'

$$\begin{array}{r} 3x + y - 3z = 1 \\ -2x - y + 2z = 1 \\ \hline x - z = 2 \\ \hline \end{array} \quad ...(4)$$

From (2) and (3) eliminate 'y'

$$\begin{array}{r} -2x - y + 2z = 1 \\ (+) \quad (+) \quad (-) \quad (-) \\ -x - y + z = 2 \\ \hline -x + z = -1 \\ \hline \end{array} \quad ...(5)$$

Solve (4) and (5)

$$\begin{array}{r} x - z = 2 \\ -x + z = -1 \\ \hline 0 = 1 \\ \hline \end{array}$$

Here we arrive at a contradiction as $0 \neq 1$.

$\therefore$ The system is inconsistent and has no solution.

---

2. (ii) $\boldsymbol{2y + z = 3(-x + 1) \,;\, -x + 3y - z = -4 \,;\, 3x + 2y + z = \frac{-1}{2}}$

$$2y + z = 3(-x + 1)$$

$$2y + z = -3x + 3$$

$$3x + 2y + z = 3 \quad ...(1)$$

$$-x + 3y - z = -4 \quad ...(2)$$

$$3x + 2y + z = \frac{-1}{2}$$

$$6x + 4y + 2z = -1 \quad ...(3)$$

From (1) and (2) eliminate 'z'

$$\begin{array}{r} 3x + 2y + z = 3 \\ -x + 3y - z = -4 \\ \hline 2x + 5y = -1 \\ \hline \end{array} \quad ...(4)$$

From (2) & (3) eliminate 'z'

$$\begin{array}{r} (3) \times 2 \Rightarrow -2x + 6y - 2z = -8 \\ (3) \Rightarrow 6x + 4y + 2z = -1 \\ \hline 4x + 10y = -9 \\ \hline \end{array} \quad ...(5)$$

Solve (4) and (5)

$$(4) \times 2 \Rightarrow 4x + 10y = -2$$

$$(-) \quad (-) \quad (+)$$

$$(5) \Rightarrow 4x + 10y = -9$$

$$0 = 7$$

Here we arrive at a contradiction as $0 \neq 1$.

$\therefore$ The system is inconsistent and has no solution.

---

2. (iii) $\dfrac{y+z}{4} = \dfrac{z+x}{3} = \dfrac{x+y}{2}$ **;** $x+y+z=27$

Let $\dfrac{y+z}{4} = \dfrac{z+x}{3}$

$$4z + 4x = 3y + 3z$$

$$4x - 3y - 3z + 4z = 0$$

$$4x - 3y + z = 0 \quad \text{...(1)}$$

Let $\dfrac{z+x}{3} = \dfrac{x+y}{2}$

$$2z + 2x = 3x + 3y$$

$$3x + 3y - 2z - 2x = 0$$

$$x + 3y - 2z = 0 \quad \text{...(2)}$$

$$x + y + z = 27 \quad \text{...(3)}$$

From (1) and (2) eliminate 'y'

$$4x - 3y + z = 0$$

$$x + 3y - 2z = 0$$

$$5x - z = 0 \quad \text{...(4)}$$

From (2) and (3) eliminate 'y'

$$x + 3y - 2z = 0$$

$$(-) \quad (-) \quad (-) \quad (-)$$

$$(3) \times 3 \Rightarrow 3x + 3y + 3z = 81$$

$$-2x - 5z = -81$$

$$2x + 5z = 81 \quad \text{...(5)}$$

Solve (4) and (5)

$$(4) \times 5 \Rightarrow 25x - 5z = 0$$

$$(5) \Rightarrow 2x + 5z = 81$$

$$27x = 81$$

$$x = \frac{81}{27}$$

$$\boxed{x = 3}$$

$$(4) \Rightarrow 5\,(3) - z = 0$$

$$\boxed{15 = Z}$$

$$(3) \Rightarrow 3 + y + 15 = 27$$

$$y + 18 = 27$$

$$y = 27 - 18$$

$$\boxed{y = 9}$$

$\therefore$ It has unique solution

$$x = 3\,;\, y = 9\,;\, z = 15$$

Type IV: Word problems

Q.No. 3, 4, 5, Example 3.4, 3.9

3. Vani, her father and her grand father have an average age of 53, one half of her grand father's age plus one-third of her father's age plus one fourth of vani's age is 65. Four years ago if vani's grandfather was four times as old as vani then how old are they all now?

Let vani's age = 'x' years

Father's age = 'y' years

grand father's age = 'z' years

Given

$$\frac{x+y+z}{3} = 53$$

$$x + y + z = 159 \quad \text{...(1)}$$

$$\frac{x}{4} + \frac{y}{3} + \frac{z}{2} = 65$$

$$\frac{3x + 4y + 6z}{12} = 65$$

$$3x + 4y + 6z = 780 \quad \text{...(2)}$$

$$z - 4 = 4\,(x - 4)$$

$$z - 4 = 4x - 16$$

$$4x - z = 12 \quad \text{...(3)}$$

From (1) and (2) eliminate 'y'

$$(1) \times \Rightarrow 4x + 4y + 4z = 636$$

$$(-) \quad (-) \quad (-) \quad (-)$$

$$(2) \Rightarrow 3x + 4y + 6z = 780$$

$$x - 2z = -144 \qquad \text{...(4)}$$

Solve (3) and (4)

$$(3) \times 2 \Rightarrow 8x - 2z = 24$$

$$(-) \quad (+) \quad (+)$$

$$(4) \Rightarrow x - 2z = -144$$

$$7x = 168$$

$$x = \frac{168}{7}$$

$$\boxed{x = 24}$$

$$(3) \Rightarrow 4\,(24) - z = 12$$

$$96 - z = 12$$

$$96 - 12 = z$$

$$\boxed{84 = z}$$

$$(1) \Rightarrow 24 + y + 84 = 159$$

$$108 + y = 159$$

$$y = 159 - 108$$

$$\boxed{y = 51}$$

$\therefore$ Age of vani = 24 years

Her father's age = 51 years

Her grandfather's age = 84 years

4. The sum of the digits of a three-digit number is 11. If the digits are reversed, the new number is 46 more than five times the former number. If the hundreds digit plus twice the tens digits is equal to the unit digit then find the original three digit number.

Let the three digit number

$$100x + 10y + z$$

Reversed number

$$100z + 10y + x$$

Given $x + y + z = 11$ ...(1)

$$100z + 10y + x = 5\,(100x + 10y + z) + 46$$

$$100z + 10y + x = 500x + 50y + 5z + 46$$

$$500x + 50y + 5z - 100z - 10y - x = -46$$

$$499x + 40y - 95z = -46 \qquad \text{...(2)}$$

$$x + 2y = z$$

$$x + 2y - z = 0 \qquad \text{...(3)}$$

From (1) and (2) eliminate 'z'

$$(1) \times 95 \Rightarrow 95x + 95y + 95z = 1045$$

$$(2) \Rightarrow 499x + 40y - 95z = -46$$

$$594x + 135y = 999$$

divide by 27; $22x + 5y = 37$ ...(4)

Solve (1) and (3)

$$x + y + z = 11$$

$$x + 2y - z = 0$$

$$2x + 3y = 11 \qquad (5)$$

Solve (4) and (5)

$$22x + 5y = 37$$

$$(-) \quad (-) \quad (-)$$

$$(5) \times 11 \Rightarrow 22x + 33y = 121$$

$$-28y = -84$$

$$y = \frac{84}{28}$$

$$\boxed{y = 3}$$

$$(5) \Rightarrow 2x + 3\,(3) = 11$$

$$2x + 9 = 11$$

$$2x = 11 - 9$$

$$2x = 2$$

$$\boxed{x = 1}$$

$$(1) \Rightarrow 1 + 3 + z = 11$$

$$4 + z = 11$$

$$z = 11 - 4$$

$$\boxed{z = 7}$$

$\therefore$ Three digit number is

$$100x + 10y + z = 100\,(1) + 10\,(3) + 7$$
$$= 100 + 30 + 7$$
$$= 137$$

5. There are 12 pieces of five, ten and twenty rupee currencies whose total value is Rs. 105. When the first 2 sorts are interchanged in their numbers its value will be increased by Rs. 20. Find the number of currencies in each sort.

Let Five rupee note $= x$

Ten rupee note $= y$

Twenty rupee note $= z$

$$\therefore \quad x + y + z = 12 \qquad \text{...(1)}$$

$$5x + 10y + 20z = 105$$

Divide by 5

$$x + 2y + 4z = 21 \qquad \text{...(2)}$$

$$10x + 5y + 20z = 105 + 20$$

$$10x + 5y + 20z = 125$$

divide by 5

$$2x + y + 4z = 25 \qquad \text{...(3)}$$

From (1) and (2) eliminate 'z'

$$(1) \times 4 \Rightarrow 4x + 4y + 4z = 48$$
$$(-)\ (-)\ (-)\ (-)$$
$$(2) \Rightarrow x + 2y + 4z = 21$$
$$3x + 2y = 27 \qquad \text{...(4)}$$

From (2) and (3) eliminate 'z'

$$x + 2y + 4z = 21$$
$$(-)\ (-)\ (-)\ (-)$$
$$2x + y + 4z = 25$$
$$-x + y = -4 \qquad \text{...(5)}$$

Solve (4) and (5)

$$(4) \Rightarrow 3x + 2y = 27$$
$$(5) \times 3 \Rightarrow -3x + 3y = -12$$
$$5y = 15$$
$$\boxed{y = 3}$$
$$(5) \Rightarrow -x + 3 = -4$$
$$-x = -4 - 3$$
$$-x = -7$$
$$\boxed{x = 7}$$
$$(1) \Rightarrow 7 + 3 + z = 12$$
$$z = 12 - 10$$
$$\boxed{z = 2}$$

$\therefore$ Number of Rs. 5 Note is 7

Number of Rs. 10 Note is 3

Number of Rs. 20 Note is 2

Example 3.4

In an interschool athletic meet, with 24 individual events, securing a total of 56 points, a first place secures 5 points, a second place secures 3 points and a third place secures 1 point. Having as many third place finishers as first an second place finishers, find how many athletes finished in each place.

Let

I place finishers $= x$

II place finishers $= y$

III place finishers$= z$

Given

$$x + y + z = 24 \qquad \text{...(1)}$$
$$5x + 3y + z = 56 \qquad \text{...(2)}$$
$$x + y = z \qquad \text{...(3)}$$

Put (3) in (1)

$$z + z = 24$$
$$2z = 24$$
$$\boxed{z = 12}$$

$(3) \Rightarrow x + y = 12$ ...(4)

From (1) and (2) eliminate 'z'

$$\begin{array}{r} x + y + z = 24 \\ (-)\ (-)\ (-)\ (-) \\ 5x + 3y + z = 56 \\ \hline -4x - 2y = -32 \\ \hline \end{array}$$

Divide by 2

$$-2x - y = -16 \quad ...(5)$$

Solve (4) and (5)

$$\begin{array}{r} x + y = 12 \\ -2x - y = -16 \\ \hline -x = -4 \\ \hline \end{array}$$

$$\boxed{x = 4}$$

$(3) \Rightarrow 4 + y = 12$

$$y = 12 - 4$$

$$\boxed{y = 8}$$

$\therefore$ Number of I place finishers = 4

Number of II place finishers = 8

Number of III place finishers = 12

Example 3.9

The sum of thrice the first number, second number and twice the third number is 5. If thrice the second number is subtracted from the sum of first number and thrice the third we get 2. If the third number is subtracted from the sum of twice the first, thrice the second we get 1. Find the numbers.

Let three numbers x, y, z respectively.

Given

$$3x + y + 2z = 5 \quad ...(1)$$

$$x + 3z - 3y = 2$$

$$x - 3y + 3z = 2 \quad ...(2)$$

$$2x + 3y - z = 1 \quad ...(3)$$

From (1) and (2) eliminate 'y'

$$\begin{array}{r} (1) \times 3 \Rightarrow 9x + 3y + 6z = 15 \\ (2) \Rightarrow x - 3y + 3z = 2 \\ \hline 10x + 9z = 17 \\ \hline \end{array} \quad ...(4)$$

From (2) and (3) eliminate 'y'

$$\begin{array}{r} x - 3y + 3z = 2 \\ 2x + 3y - z = 1 \\ \hline 3x + 2z = 3 \\ \hline \end{array} \quad ...(5)$$

Solve (4) and (5)

$$\begin{array}{r} (4) \times 2 \Rightarrow 20x + 18z = 34 \\ (-)\ (-)\ (-) \\ (5) \times 9 \Rightarrow 27x + 18z = 27 \\ \hline -7x = 7 \\ \hline \end{array}$$

$$x = \frac{7}{-7}$$

$$\boxed{x = -1}$$

$(5) \Rightarrow 3(-1) + 2z = 3$

$$-3 + 2z = 3$$

$$2z = 3 + 3$$

$$2z = 6$$

$$z = \frac{6}{2}$$

$$\boxed{z = 3}$$

$(3) \Rightarrow 2(-1) + 3y - 3 = 1$

$$-2 + 3y - 3 = 1$$

$$3y = 1 + 2 + 3$$

$$y = \frac{6}{3}$$

$$\boxed{y = 2}$$

$\therefore$ The numbers are $-1, 2, 3$

Exercise 3.2

Type I: GCD of two given polynomials by division algorithm

Q.No. 1.(i) (ii) (iii) (iv), Example 3.10, 3.11

1. Find the GCD of the given polynomials

(i) $\mathbf{x^4 + 3x^3 - x - 3 \,;\, x^3 + x^2 - 5x + 3}$

$$f(x) = x^4 + 3x^3 - x - 3$$

$$g(x) = x^3 + x^2 - 5x + 3$$

Here degree of $f(x) >$ degree of $g(x)$

$$\therefore \quad f(x) \div g(x)$$

$$\begin{array}{r|l}
 & x + 2 \\ \hline
x^3 + x^2 - 5x + 3 & x^4 + 3x^3 + 0x^2 - x - 3 \\
 & (-) \ (-) \ (+) \ (-) \\
 & x^4 + 1x^3 - 5x^2 + 3x \\ \hline
 & \quad 2x^3 + 5x^2 - 4x - 3 \\
 & \quad (-) \ (-) \ (+) \ (-) \\
 & \quad 2x^3 + 2x^2 - 10x + 6 \\ \hline
 & \qquad 3x^2 + 6x - 9
\end{array}$$

$\therefore \quad R = 3\,(x^2 + 2x - 3)$; Note that 3 is not a division of $g(x)$.

So let $R(x) = x^2 + 2x - 3$ only.

$$\begin{array}{r|l}
 & x - 1 \\ \hline
x^2 + 2x - 3 & x^3 + x^2 - 5x + 3 \\
 & (-) \ (-) \ (+) \\
 & x^3 + 2x^2 - 3x \\ \hline
 & \quad - x^2 - 2x + 3 \\
 & \quad (+) \ (+) \ (-) \\
 & \quad - x - 2x + 3 \\ \hline
 & \qquad 0
\end{array}$$

$$\therefore \quad \text{GCD} = (x^2 + 2x - 3)$$

(ii) $\mathbf{x^4 - 1,\, x^3 - 11x^2 + x - 11}$

$$f(x) = x^4 + 0x^3 + 0x^2 + 0x - 1$$

$$g(x) = x^3 - 11x^2 + x - 11$$

Here $\deg f(x) > \deg g(x)$

$$\therefore \quad f(x) \div g(x)$$

$$\begin{array}{r|l}
 & x + 11 \\ \hline
x^3 - 11x^2 + x - 11 & x^4 + 0x^3 + 0x^2 + 0x - 1 \\
 & (-) \ (+) \ (-) \ (+) \\
 & x^4 - 11x^3 + 1x^2 - 11x \\ \hline
 & \quad 11x^3 - x^2 + 11x - 1 \\
 & \quad (-) \ (+) \ (-) \ (+) \\
 & \quad 11x^3 - 121x^2 + 11x - 121 \\ \hline
 & \qquad 120x^2 + 0x + 120
\end{array}$$

$R(x) = 120\,(x^2 + 0x + 1)$ Note that 120 is not a division of $g(x)$

Now $g(x) \div R(x)$

$$\begin{array}{r|l}
 & x - 11 \\ \hline
x^2 + 0x + 1 & x^3 - 11x^2 + x - 11 \\
 & (-) \ (-) \ (-) \\
 & x + 0x + x \\ \hline
 & \quad - 11x^2 + 0x - 11 \\
 & \quad (+) \ (-) \ (+) \\
 & \quad - 11x + 0x - 11 \\ \hline
 & \qquad 0
\end{array}$$

$$\therefore \text{GCD} = (x^2 + 1)$$

(iii) $\mathbf{3x^4 + 6x^3 - 12x^2 - 24x,}$
$\mathbf{4x^4 + 14x^3 + 8x^2 - 8x}$

$$f(x) = 3x^4 + 6x^3 - 12x^2 - 24x$$

$$= 3x\,[x^3 + 2x^2 - 4x - 8]$$

$$g(x) = 4x^4 + 14x^3 + 8x^2 - 8x$$

$$= 2x\,(2x^3 + 7x^2 + 4x - 4)$$

Here $g(x) \div f(x)$

$$
\begin{array}{r|l}
 & \quad 2 \\ \hline
x^3+2x^2-4x-8 & 2x^3+7x^2+4x-4 \\
 & (-)\ (-)\ (+)\ (+) \\
 & 2x^3+4x^2-8x-16 \\ \hline
 & \quad 3x^2+12x+12
\end{array}
$$

$R(x) = 3(x^2+4x+4)$

Now $f(x) \div R(x)$

$$
\begin{array}{r|l}
 & x-2 \\ \hline
x^2+4x+4 & x^3+2x^2-4x-8 \\
 & (-)\ (-)\ (-) \\
 & x^3+4x^2+4x \\ \hline
 & -2x^2-8x-8 \\
 & (+)\ (+)\ (+) \\
 & -2x-8x-8 \\ \hline
 & \quad 0
\end{array}
$$

$\therefore$ GCD $= x(x^2+4x+4)$

(iv) $\mathbf{3x^3+3x^2+3x+3\ ;\ 6x^3+12x^2+6x+12}$

$f(x) = 3x^2+3x^2+3x+3$

$= 3(x^3+x^2+x+1)$

$g(x) = 6x^3+12x^2+6x+12$

$= 6(x^3+2x^2+x+2)$

Here $g(x) \div f(x)$

[Also we can take $f(x) \div g(x)$ since $\deg f(x) = \deg g(x)$]

$$
\begin{array}{r|l}
 & 1 \\ \hline
x^3+x^2+x+1 & \\
 & (-)\ (-)\ (-)\ (-) \\
 & x^3+x^2+x+1 \\ \hline
 & \quad x^2+0x+1
\end{array}
$$

$R(x) = x^2+0x+1$

Now $f(x) \div R(x)$

$$
\begin{array}{r|l}
 & x+1 \\ \hline
x^2+0x+1 & x^3+x^2+x+1 \\
 & (-)\ (-)\ (-) \\
 & x+0x+x \\ \hline
 & \quad x^2+0x+1 \\
 & \quad (-)\ (-)\ (-) \\
 & \quad x^2+0x+1 \\ \hline
 & \quad 0
\end{array}
$$

GCD of leading co-efficients 3 and 6 is 3

$\therefore$ GCD $= 3(x^2+1)$

Example 3.10

Find the GCD of the polynomials $\boldsymbol{x^2+x^2-x+2}$ ***and*** $\boldsymbol{2x^3-5x^2+5x-3}$

$f(x) = x^3+x^2-x+2$

$g(x) = 2x^3-5x^2+5x-3$

Here $g(x) \div f(x)$

$$
\begin{array}{r|l}
 & 2 \\ \hline
x^3+x^2-x+2 & 2x^3-5x^2+5x-3 \\
 & (-)\ (-)\ (+)\ (-) \\
 & 2x^3+2x^2-2x+4 \\ \hline
 & \quad -7x^2+7x-7
\end{array}
$$

$R(x) = -7(x^2-x+1)$

Note that -7 is not a divisor of $f(x)$

Now $f(x) \div R(x)$

$$
\begin{array}{r|l}
 & x+2 \\ \hline
x^2-x+1 & x^3+x^2-x+2 \\
 & (-)\ (+)\ (-) \\
 & x^3-x^2+x \\ \hline
 & \quad 2x^2-2x+2 \\
 & \quad (-)\ (+)\ (-) \\
 & \quad 2x^2-2x+2 \\ \hline
 & \quad 0
\end{array}
$$

$\therefore\ GCD = (x^2-x+1)$

Example 3.11

Find the GCD of $6x^3 - 30x^2 + 60x - 48$ *and* $3x^3 - 12x^2 + 21x - 18$

$$f(x) = 6x^3 - 30x^2 + 60x - 48$$
$$= 6(x^3 - 5x^2 + 10x - 8)$$
$$g(x) = 3x^3 - 12x^2 + 21x - 18$$
$$= 3(x^3 - 4x^2 + 7x - 6)$$

Here $f(x) \div g(x)$

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3 - 4x^2 + 7x - 6 & x^3 - 5x^2 + 10x - 8 \\ & (-)\ (+)\ (-)\ (+) \\ & x^3 - 4x^2 + 7x - 6 \\ \hline & \quad -x^2 + 3x - 2 \end{array}$$

$$R(x) = -[x^2 - 3x + 2]$$

Now $g(x) \div R(x)$

$$\begin{array}{r|l} & \quad x - 1 \\ \hline x^2 - 3x + 2 & x^3 - 4x^2 + 7x - 6 \\ & (-)\ (+)\ (-) \\ & x^3 - 3x^2 + 2x \\ \hline & \quad -x^2 + 5x - 6 \\ & \quad (+)\ (-)\ (+) \\ & \quad -x^2 + 3x - 2 \\ \hline & \quad\quad 2x - 4 \end{array}$$

$R(x) = 2(x - 2)$; Remainder $\neq 0$ so again divide

$$\begin{array}{r|l} & \quad x - 1 \\ \hline x - 2 & x^2 - 3x + 2 \\ & (-)\ (+) \\ & x^2 - 2x \\ \hline & \quad -x + 2 \\ & \quad -x + 2 \\ \hline & \quad\quad 0 \end{array}$$

$\therefore$ GCD of leading co-efficients 3 and 6 is 3.

$\therefore$ GCD $= 3(x - 2)$

Type II: LCM by factorization method

Q.No. 2(i) to (vi), Example 3.12(i) to (iv)

2. Find the LCM of the given expressions

(i) $4x^2 y, 8x^3 y^2$

LCM $= 8x^3 y^2$

$$\begin{array}{r|l} 2 & 4,\ 8 \\ \hline 2 & 2,\ 4 \\ \hline & 1,\ 2 \end{array}$$

LCM $= 2 \times 2 \times 2$
$= 8$

(ii) $-9a^3 b^2, 12a^2 b^2 c$

LCM $= -36a^3 b^2 c$

$$\begin{array}{r|l} 3 & 9,\ 12 \\ \hline & 3,\ 4 \end{array}$$

LCM $= 3 \times 3 \times 4$
$= 36$

(iii) $16m, -12m^2 n^2, 8n^2$

LCM $= -48m^2 n^2$

$$\begin{array}{r|l} 2 & 16,\ 12,\ 8 \\ \hline 2 & 8,\ 6,\ 4 \\ \hline 2 & 4,\ 3,\ 2 \\ \hline & 2,\ 3,\ 1 \end{array}$$

LCM $= 2 \times 2 \times 2 \times 2 \times 3$
$= 48$

(iv) $P^2 - 3P + 2$; $P^2 - 4$

- $P^2 - 3P + 2 = (P - 1)(P - 2)$
- $P^2 - 4 = (P + 2)(P - 2)$

$\therefore$ LCM $= (P - 2)(P - 1)(P + 2)$

(v) $2x^2 - 5x - 3$; $4x^2 - 36$

- $2x^2 - 5x - 3$
  $= (x - 3)(2x + 1)$
- $4x^2 - 36$
  $= 4(x^2 - 9)$
  $= 4(x + 3)(x - 3)$

$\therefore$ LCM $= 4(x - 3)(x + 3)(2x + 1)$

$$-6$$
$$\begin{array}{cc} -6 & 1 \\ \frac{-6}{2} & \frac{1}{2} \\ -3 & \frac{1}{2} \end{array}$$

(vi) $(2x^2 - 3xy)^2$; $(4x - 6y)^3$; $8x^3 - 27y^3$

- $(2x^2 - 3xy)^2$
  $= [x(2x - 3y)]^2$
  $= x^2 (2x - 3y)^2$

- $(4x-6y)^3$

  $= [2\,(2x-3y)]^3$

  $= 2^3\,(2x-3y)^3$

  $= 8\,(2x-3y)^3$

- $8x^3-27y^3$

  $= (2x)^3-(3y)^3$

  $= (2x-3y)\,(4x^2+6xy+9y^2)$

  $\therefore$ LCM $= 8x^2\,(2x-3y)^3\,(4x^2+6xy+9y^2)$

Example 3.12

Find the LCM of the following

(i) $8x^4y^2,\ 48x^2y^4$

LCM $= 48x^4y^4$

(ii) $5x-10,\ 5x^2-20$

- $5x-10 = 5\,(x-2)$
- $5x^2-20 = 5\,(x^2-4)$

  $= 5\,(x+2)\,(x-2)$

LCM $= 5\,(x-2)\,(x+2)$

(iii) $x^4-1,\ x^2-2x+1$

- $x^4-1 = (x^2+1)\,(x^2-1)$

  $= (x^2+1)\,(x+1)\,(x-1)$

- $x^2-2x+1 = (x-1)^2$

LCM $= (x^2+1)\,(x+1)\,(x-1)^2$

(iv) $x^3-27,\ (x-3)^2,\ x^2-9$

- $x^3-27 = x^3-3^3$

  $= (x-3)\,(x^2+3x+9)$

- $(x-3)^2$
- $x^2-9 = (x+3)\,(x-3)$

LCM $= (x-3)^2\,(x+3)\,(x^2+3x+9)$

Exercise 3.3

Key points

Relationship between LCM and GCD

Product of two polynomials = Product of their LCM and GCD

$$\boxed{f\,(x)\;g\,(x) = \text{LCM}\times\text{GCD}}$$

Note:

- $f\,(x) = \dfrac{\text{LCM}\times\text{GCD}}{g\,(x)}$
- $\text{LCM} = \dfrac{f\,(x)\cdot g\,(x)}{\text{GCD}}$

Type I: Verify $f\,(x)\times g\,(x) = \text{LCM}\times\text{GCD}$

Q.No:1. (i) (ii) (iii)

(i) $21x^2y,\ 35xy^2$

$$\begin{array}{r|l} 7 & 21,\ 35 \\ \hline & 3,\ 5 \end{array}$$

LCM $= 7\times3\times5$

$= 105$

LCM $= 105x^2y^2$

GCD $= 7xy$

Verification

$f\,(x)\times g\,(x) = \text{LCM}\times\text{GCD}$

$(21x^2y)\,(35xy^2) = (105x^2y^2)\,(7xy)$

$735x^3y^3 = 735\,x^3y^3$

Hence verified.

(ii) $(x^3-1)\,(x+1),\ (x^3+1)$

- $(x^3-1)\,(x+1)$

  $= (x-1)\,(x^2+x+1)\,(x+1)$

- x^3+1

  $= (x+1)\,(x^2-x+1)$

LCM $= (x+1)\,(x-1)\,(x^2+x+1)\,(x^2-x+1)$

$= (x+1)\,(x^2-x+1)\,(x-1)\,(x^2+x+1)$

$= (x^3+1)\,(x^3-1)$

(ii) Multiply $\dfrac{x^4 b^2}{x-1}$ by $\dfrac{x^2-1}{a^4 b^3}$

$$=\frac{x^4 b^2}{x-1}\times\frac{x^2-1}{a^4 b^3}$$

$$=\frac{x^4 b^2}{x-1}\times\frac{(x+1)(x-1)}{a^4 b^3}$$

$$=\frac{x^4(x+1)}{a^4 b}$$

Type II: Division of rational expressions

Q.No. 3(i) (ii) (iii) (iv), Example 3.16 (i) (ii) (iii), Q.No. 4,5

3. Simplify:

(i) $\dfrac{2a^2+5a+3}{2a^2+7a+b}\div\dfrac{a^2+6a+5}{-5a^2-35a-50}$

Let $\dfrac{2a^2+5a+3}{2a^2+7a+6}\times\dfrac{-5a^2-35a-50}{a^2+6a+5}$

- $2a^2+5a+3$

  $=(a+1)(2a+3)$

- $2a^2+7a+6$

  $=(a+2)(2a+3)$

- $-5a^2-35a-50$

  $=-5(a^2+7a+10)$

  $=-5(a+2)(a+5)$

- a^2+6a+6

  $=(a+1)(a+5)$

6: 2, 3; $\frac{2}{2}$, $\frac{3}{2}$; 1, $\frac{3}{2}$

12: 4, 3; $\frac{4}{2}$, $\frac{3}{2}$; 2, $\frac{3}{2}$

$$\therefore\ \frac{2a^2+5a+3}{2a^2+7a+6}\times\frac{-5a^2-35a-50}{a^2+6a+5}$$

$$=\frac{(a+1)(2a+3)}{(a+2)(2a+3)}\times\frac{-5(a+2)(a+5)}{(a+1)(a+5)}$$

$$=-5$$

(ii) $\dfrac{b^2+3b-28}{b^2+4b+4}\div\dfrac{b^2-49}{b^2-5b-14}$

Let $\dfrac{b^2+3b-28}{b^2+4b+4}\times\dfrac{b^2-5b-14}{b^2-49}$

$$=\frac{(b+7)(b-4)}{(b+2)^2}\times\frac{(b-7)(b+2)}{(b+7)(b-7)}$$

$$=\frac{b-4}{b+2}$$

(iii) $\dfrac{x+2}{4y}\div\dfrac{x^2-x-6}{12y^2}$

Let $\dfrac{x+2}{4y}+\dfrac{12y^2}{x^2-x-6}$

$$=\frac{x+2}{4y}\times\frac{12y^2}{(x-3)(x+2)}$$

$$=\frac{3y}{x-3}$$

(iv) $\dfrac{12t^2-22t+8}{3t}\div\dfrac{3t^2+2t-8}{2t^2+4t}$

Let $\dfrac{12t^2-22t+8}{3t}\times\dfrac{2t^2+4t}{3t^2+2t-8}$

- $12t^2-22t+8$

  $=2(6t^2-11t+4)$

  $=2(2t-1)(3t-4)$

- $2t^2+4t=2t(t+2)$

- $3t^2+2t-8$

  $=(t+2)(3t-4)$

24: -3, -8; $\frac{-3}{6}$, $\frac{-8}{6}$; $\frac{-1}{2}$, $\frac{-4}{3}$

-24: 6, -4; $\frac{6}{3}$, $\frac{-4}{3}$; 2, $\frac{-4}{3}$

$$\therefore\ \frac{12t^2-22t+8}{3t}\times\frac{2t^2+4t}{3t^2+2t-8}$$

$$=\frac{2(2t-1)(3t-4)}{3t}\times\frac{2t(t+2)}{(t+2)(3t-4)}$$

$$=\frac{4}{3}(2t-1)$$

Example 3.16

Find (i) $\dfrac{14x^4}{y} \div \dfrac{7x}{3y^4}$

Let $\dfrac{14x^4}{y} \times \dfrac{3y^4}{7x}$

$= 2x^3 \times 3y^3$

$= 6x^3\,y^3$

(ii) $\dfrac{x^2-16}{x+4} \div \dfrac{x-4}{x+4}$

Let $\dfrac{x^2-16}{x+4} \times \dfrac{x+4}{x-4}$

$= \dfrac{(x+4)\,(x-4)}{x+4} \times \dfrac{x+4}{x-4}$

$= x+4$

(iii) $\dfrac{16x^2-2x-3}{3x^2-2x-1} \div \dfrac{8x^2+11x+3}{3x^2-11x-4}$

Let $\dfrac{16x^2-2x-3}{3x^2-2x-1} \times \dfrac{3x^2-11x-4}{8x^2+11x+3}$

- $16x^2-2x-3$

  $= (2x-1)\,(8x+3)$

$$\begin{array}{cc} \multicolumn{2}{c}{-48} \\ -8 & 6 \\ \dfrac{-8}{16} & \dfrac{6}{16} \\ \dfrac{-1}{2} & \dfrac{3}{8} \end{array}$$

- $3x^2-2x-1$

  $= (x-1)\,(3x+1)$

$$\begin{array}{cc} \multicolumn{2}{c}{-3} \\ -3 & 1 \\ \dfrac{-3}{3} & \dfrac{1}{3} \\ -1 & \dfrac{1}{3} \end{array}$$

- $3x^2-11x-4$

  $= (x-4)\,(3x+1)$

$$\begin{array}{cc} \multicolumn{2}{c}{-12} \\ -12 & 1 \\ \dfrac{-12}{3} & \dfrac{1}{3} \\ -4 & \dfrac{1}{3} \end{array}$$

- $8x^2+11x+3$

  $= (x+1)\,(8x+3)$

$$\begin{array}{cc} \multicolumn{2}{c}{24} \\ 8 & 3 \\ \dfrac{8}{8} & \dfrac{3}{8} \\ 1 & \dfrac{3}{8} \end{array}$$

$\therefore \quad \dfrac{(2x-1)\,(8x+3)}{(x-1)\,(3x+1)} \times \dfrac{(x-4)\,(3x+1)}{(x+1)\,(8x+3)}$

$= \dfrac{(2x-1)\,(x-4)}{(x-1)\,(x+1)}$

$= \dfrac{2x^2-8x-x+4}{x^2-1}$

$= \dfrac{2x^2-9x+4}{x^2-1}$

---

4. If $x = \dfrac{a^2+3a-4}{3a^2-3}$ and $y = \dfrac{a^2+2a-8}{2a^2-2a-4}$ find the value of $x^2\,y^{-2}$

$x = \dfrac{a^2+3a-4}{3a^2-3}$

$= \dfrac{(a+4)\,(a-1)}{3\,(a^2-1)}$

$= \dfrac{(a+4)\,(a-1)}{3\,(a+1)\,(a-1)}$

$= \dfrac{a+4}{3\,(a+1)}$

$y = \dfrac{a^2+2a-8}{2a^2-2a-4}$

$= \dfrac{(a+4)\,(a-2)}{2\,(a^2-a-2)}$

$$= \frac{(a+4)(a-2)}{2(a-2)(a+1)}$$

$$= \frac{a+4}{2(a+1)}$$

Now $x^2 y^{-2} = \frac{x^2}{y^2} = \left(\frac{x}{y}\right)^2$

$$= \left[\frac{a+4}{3(a+1)} \div \frac{a+4}{2(a+1)}\right]^2$$

$$= \left[\frac{a+4}{3(a+1)} \times \frac{2(a+1)}{a+4}\right]^2$$

$$= \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

5. If a polynomial $p(x) = x^2 - 5x - 14$ is divided by another polynomial $q(x)$ we get $\frac{x-7}{x+2}$, find $q(x)$

Given

$$p(x) \div q(x) = \frac{x-7}{x+2}$$

$$\frac{x^2 - 5x - 14}{q(x)} = \frac{x-7}{x+2}$$

$$\frac{(x-7)(x+2)}{q(x)} = \frac{x-7}{x+2}$$

$$q(x) = (x+2)^2$$

$$q(x) = x^2 + 4x + 4$$

Exercise 3.6

Key points

1. Addition and substraction of rational expressions with like denominators. - (No LCM directly add or sub)
2. Addition and subtraction of rational expressions with unlike denominators.
   - Take LCM of denominator then add or sub.

Type I: Rational expression with like denominator

Q.No. 1(i) (iii), 2(i), Example 3.17, 6

1. Simplify (i) $\frac{x(x+1)}{x-2} + \frac{x(1-x)}{x-2}$

$$= \frac{x(x+1)}{x-2} + \frac{x(1-x)}{x-2}$$

$$= \frac{x(x+1) + x(1-x)}{x-2}$$

$$= \frac{x^2 + x + x - x^2}{x-2}$$

$$= \frac{2x}{x-2}$$

(iii) $\frac{x^3}{x-y} + \frac{y^3}{y-x}$

$$= \frac{x^3}{x-y} + \frac{y^3}{y-x}$$

$$= \frac{x^3}{x-y} + \frac{y^3}{-(x-y)}$$

$$= \frac{x^3 - y^3}{x-y}$$

$$= \frac{(x-y)(x^2 + xy + y^2)}{x-y}$$

$$= x^2 + xy + y^2$$

2. Simplify (i)

$$\frac{(2x+1)(x-2)}{x-4} - \frac{(2x^2 - 5x + 2)}{x-4}$$

$$= \frac{(2x+1)(x-2)}{x-4} - \frac{(2x^2 - 5x + 2)}{x-4}$$

$$= \frac{(2x+1)(x-2) - (2x^2 - 5x + 2)}{x-4}$$

$$= \frac{2x^2 - 4x + x - 2 - 2x^2 + 5x - 2}{x-4}$$

$$= \frac{2x-4}{x-4}$$

$$= \frac{2(x-2)}{x-4}$$

Example 3.17

Find $\dfrac{x^2+20x+36}{x^2-3x-28} - \dfrac{x^2+12x+4}{x^2-3x-28}$

$$= \frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28}$$

$$= \frac{x^2+20x+36-x^2-12x-4}{x^2-3x-28}$$

$$= \frac{8x+32}{x^2-3x-28}$$

$$= \frac{8(x+4)}{(x-7)(x+4)}$$

$$= \frac{8}{x-7}$$

6. If $A = \dfrac{x}{x+1}, B = \dfrac{1}{x+1}$ **prove that** $\dfrac{(A+B)^2+(A-B)^2}{A+B} = \dfrac{2(x^2+1)}{x(x+1)^2}$

Given $A = \dfrac{x}{x+1}, B = \dfrac{1}{x+1}$

$$A+B = \frac{x}{x+1} + \frac{1}{x+1}$$

$$= \frac{x+1}{x+1}$$

$$\boxed{A+B=1}$$

$$A-B = \frac{x}{x+1} - \frac{1}{x+1}$$

$$= \frac{x-1}{x+1}$$

$$(A-B)^2 = \left(\frac{x-1}{x+1}\right)^2 \text{é}$$

$$= \frac{(x-1)^2}{(x+1)^2}$$

$$A \div B = \frac{x}{x+1} \times \frac{x+1}{1}$$

$$= x$$

$\therefore$ **LHS**

$$\frac{(A+B)^2+(A-B)^2}{A \div B)}$$

$$= \frac{(1)^2 + \dfrac{(x-1)^2}{(x+1)^2}}{x}$$

$$= \frac{(x+1)^2+(x-1)^2}{(x+1)^2} \times \frac{1}{x}$$

$$= \frac{x^2+2x+1+x^2-2x+1}{x(x+1)^2}$$

$$= \frac{2x^2+2}{x(x+1)^2}$$

$$= \frac{2(x^2+1)}{x(x+1)^2} \quad \text{RHS}$$

Type II: Rational expressions with unlike denominator

Q.No. 1(ii), 2(ii), 3, 4, 5, Example 3.18

1. Simplify (ii) $\dfrac{x+2}{x+3} + \dfrac{x-1}{x-2}$

$$= \frac{x+2}{x+3} + \frac{x-1}{x-2}$$

$$= \frac{(x+2)(x-2)+(x-1)(x+3)}{(x+3)(x-2)}$$

$$= \frac{x^2-4+x^2+2x-3}{x^2+x-6}$$

$$= \frac{2x^2+2x-7}{x^2+x-6}$$

2. Simplify (ii) $\dfrac{4x}{x^2-1}-\dfrac{x+1}{x-1}$

$$=\frac{4x}{x^2-1}-\frac{x+1}{x-1}$$

$$=\frac{4x}{(x+1)(x-1)}-\frac{x+1}{x-1}$$

$$=\frac{4x-(x+1)(x+1)}{(x+1)(x-1)}$$

$$=\frac{4x-(x^2+2x+1)}{(x+1)(x-1)}$$

$$=\frac{4x-x^2-2x-1}{(x+1)(x-1)}$$

$$=\frac{-x^2+2x-1}{(x+1)(x-1)}$$

$$=\frac{-[x^2-2x+1]}{(x+1)(x-1)}$$

$$=\frac{-(x-1)^2}{(x+1)(x-1)}$$

$$=\frac{-(x-1)}{x+1}$$

$$=\frac{1-x}{1+x}$$

---

3. Subtract $\dfrac{1}{x^2+2}$ from $\dfrac{2x^3+x^2+3}{(x^2+2)^2}$

$$=\frac{2x^3+x^2+3}{(x^2+2)^2}-\frac{1}{x^2+2}$$

$$=\frac{2x^3+x^2+3-(x^2+2)}{(x^2+2)^2}$$

$$=\frac{2x^3+x^2+3-x^2-2}{(x^2+2)^2}$$

$$=\frac{2x^3+1}{(x^2+2)^2}$$

---

4. Which rational expression should be subtracted from $\dfrac{x^2+6x+8}{x^3+8}$ to get $\dfrac{3}{x^2-2x+4}$

Let the expression be $P(x)$

Given

$$\frac{x^2+6x+8}{x^3+8}-p(x)=\frac{3}{x^2-2x+4}$$

$$\Rightarrow \frac{x^2+6x+8}{x^3+8}-\frac{3}{x^2-2x+4}=p(x)$$

$$\frac{(x+2)(x+4)}{(x+2)(x^2-2x+4)}-\frac{3}{x^2-2x+4}=p(x)$$

$$\frac{x+4-3}{x^2-2x+4}=p(x)$$

$$\frac{x+1}{x^2-2x+4}=p(x)$$

---

5. If $A=\dfrac{2x+1}{2x-1}$; $B=\dfrac{2x-1}{2x+1}$ find $\dfrac{1}{A-B}-\dfrac{2B}{A^2-B^2}$

Let $\dfrac{1}{A-B}-\dfrac{2B}{A^2-B^2}$

$$=\frac{1}{A-B}-\frac{2B}{(A+B)(A-B)}$$

$$=\frac{A+B-2B}{(A+B)(A-B)}$$

$$=\frac{A-B}{(A+B)(A-B)}$$

$$=\frac{1}{A+B}$$

Put $A=\dfrac{2x+1}{2x-1}, B=\dfrac{2x-1}{2x+1}$

$$A+B=\frac{2x+1}{2x-1}+\frac{2x-1}{2x+1}$$

(ii) $(6x^2+x-1)(3x^2+2x-1)(2x^2+3x+1)$

Here

- $6x^2+x-1=(2x+1)(3x-1)$

$$\begin{array}{cc} \multicolumn{2}{c}{-6} \\ 3 & -2 \\ \frac{3}{6} & \frac{-2}{6} \\ \frac{1}{2} & \frac{-1}{3} \end{array}$$

- $3x^2+2x-1=(x+1)(3x-1)$

$$\begin{array}{cc} \multicolumn{2}{c}{-3} \\ 3 & -1 \\ \frac{3}{3} & \frac{-1}{3} \\ 1 & \frac{-1}{3} \end{array}$$

- $2x^2+3x+1=(x+1)(2x+1)$

$$\begin{array}{cc} \multicolumn{2}{c}{2} \\ 2 & 1 \\ \frac{2}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} \end{array}$$

$\therefore\ \sqrt{(6x^2+x-1)(3x^2+2x-1)(2x^2+3x+1)}$

$=\sqrt{(2x+1)(3x-1)(x+1)(3x-1)(x+1)(2x+1)}$

$=|(x+1)(2x+1)(3x-1)|$

---

2. (iv) $(4x^2-9x+2)(7x^2-13x-2)$
$(28x^2-3x-1)$

Here

- $4x^2-9x+2$

  $=(x-2)(4x-1)$

$$\begin{array}{cc} \multicolumn{2}{c}{8} \\ -8 & -1 \\ \frac{-8}{4} & \frac{-1}{4} \\ -2 & \frac{-1}{4} \end{array}$$

- $7x^2-13x-2$

  $=(x-2)(7x+1)$

$$\begin{array}{cc} \multicolumn{2}{c}{-14} \\ -14 & 1 \\ \frac{-14}{7} & \frac{1}{7} \\ -2 & \frac{1}{7} \end{array}$$

- $28x^2-3x-1$

  $=(4x-1)(7x+1)$

$$\begin{array}{cc} \multicolumn{2}{c}{-28} \\ -7 & 4 \\ \frac{-7}{28} & \frac{4}{28} \\ \frac{-1}{4} & \frac{1}{7} \end{array}$$

$\therefore\ \sqrt{\sqrt{(4x^2-9x+2)(7x^2-13x-2)(28x^2-3x-1)}}$

$=\sqrt{(x-2)(4x-1)(x-2)(7x+1)(4x-1)(7x+1)}$

$=|(x-2)(4x-1)(7x+1)|$

(v) $\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)$
$\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)$

Here

- $2x^2+\frac{17}{6}x+1$

  $=\frac{1}{6}[12x^2+17x+6]$

  $=\frac{1}{6}(4x+3)(3x+2)$

$$\begin{array}{cc} \multicolumn{2}{c}{72} \\ 9 & 8 \\ \frac{9}{12} & \frac{8}{12} \\ \frac{3}{4} & \frac{2}{3} \end{array}$$

- $\frac{3}{2}x^2+4x+2$

  $=\frac{1}{2}[3x^2+8x+4]$

  $=\frac{1}{2}(3x+2)(x+2)$

$$\begin{array}{cc} \multicolumn{2}{c}{12} \\ 2 & 6 \\ \frac{2}{3} & \frac{6}{3} \\ \frac{2}{3} & 2 \end{array}$$

- $\frac{4}{3}x^2+\frac{11}{3}x+2$

  $=\frac{1}{3}[4x^2+11x+6]$

  $=\frac{1}{3}(4x+3)(x+2)$

$$\begin{array}{cc} \multicolumn{2}{c}{24} \\ 3 & 8 \\ \frac{3}{4} & \frac{8}{4} \\ \frac{3}{4} & 2 \end{array}$$

$$\therefore \sqrt{\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)}$$

$$= \sqrt{\frac{1}{6}\times\frac{1}{2}\times\frac{1}{3}\times(4x+3)\,(3x+2)\,(3x+2)\,(4x+3)\,(x+2)}$$

$$= \frac{1}{6}\,|\,(x+2)\,(3x+2)\,(4x+3)\,|$$

Example 3.20

(iii) $[(\sqrt{15}\,x^2+(\sqrt{3}+\sqrt{10})\,x+\sqrt{2}]\;[\sqrt{5}\,x^2+(2\sqrt{5}+1)\,x+2]\;[\sqrt{3}\,x^2+(\sqrt{2}+2\sqrt{3})\,x+2\sqrt{2}]$

- $\sqrt{15}\,x^2+(\sqrt{3}+\sqrt{10})\,x+\sqrt{2}$

  $= \sqrt{15}\,x^2+\sqrt{3}\,x+\sqrt{10}\,x+\sqrt{2}$

  $= \sqrt{3}\,x\,(\sqrt{5}\,x+1)+\sqrt{2}\,(\sqrt{5}\,x+1)$

  $= (\sqrt{5}\,x+1)\,(\sqrt{3}\,x+\sqrt{2})$

- $\sqrt{5}\,x^2+(2\sqrt{5}+1)\,x+2$

  $= \sqrt{5}\,x^2+2\sqrt{5}\,x+x+2$

  $= \sqrt{5}\,x\,(x+2)+1\,(x+2)$

  $= (x+2)\,(\sqrt{5}\,x+1)$

- $\sqrt{3}\,x^2+(\sqrt{2}+2\sqrt{3}\,x+2\sqrt{2}$

  $= \sqrt{3}\,x^2+\sqrt{2}\,x+2\sqrt{3}\,x+2\sqrt{2}$

  $= x\,(\sqrt{3}\,x+\sqrt{2})+2\,(\sqrt{3}\,x+\sqrt{2})$

  $= (\sqrt{3}\,x+\sqrt{2})\,(x+2)$

$$\therefore \sqrt{(\sqrt{15}\,x^2+(\sqrt{3}+\sqrt{10}\,x+\sqrt{2})\,(\sqrt{5}\,x^2+(2\sqrt{5}+1)\,x+2)\,(\sqrt{3}\,x^2+(\sqrt{2}+2\sqrt{3}\,x+2\sqrt{2})}$$

$$= \sqrt{(\sqrt{5}\,x+1)\,(\sqrt{3}\,x+\sqrt{2})\,(\sqrt{5}\,x+1)\,(x+2)\,(x+2)\,(\sqrt{3}\,x+\sqrt{2})}$$

$$= |\,(x+2)\,(\sqrt{5}\,x+1)\,(\sqrt{3}\,x+\sqrt{2}\,|$$

Exercise 3.8

Square root by long division method.

Type I: Find the square root

Q.No. 1(i) (ii) (iii) (iv), Example 3.21, 3.22, Q.No. 2

1. Find the square root of the following polynomials by division method.

(i) $x^4-12x^3+42x^2-36x+7$

$$\begin{array}{r|l}
 & x^2-6x+3 \\
\hline
x^2 & x^4-12x^3+42x^2-36x+9 \\
 & (-) \\
 & x^4 \\
\hline
2x^2-6x & -12x^3+42x^2 \\
 & \quad(+)\qquad(-) \\
 & -12x^3+36x^2 \\
\hline
2x^2-12x+3 & \qquad 6x^2-36x+9 \\
 & \qquad(-)\;(+)\;(-) \\
 & \qquad 6x^2-36x+9 \\
\hline
 & \qquad\quad 0
\end{array}$$

$$\therefore \sqrt{x^4-12x^3+42x^2-36x+9} = |\,x^2-6x+3\,|$$

(ii) $37x^2-28x^3+4x^4+42x+9$

Rearrange the given polynomial as per degree in ascending or descending order

$\therefore\; 4x^4-28x^3+37x^2+42x+9$

$$\begin{array}{r|l}
 & 2x^2-7x-3 \\
\hline
2x^2 & 4x^4-28x^3+37x^2+42x+9 \\
 & (-) \\
 & 4x^4 \\
\hline
4x^2-7x & -28x^2+37x^2 \\
 & \quad(+)\;(-) \\
 & -28x^3+49x^2 \\
\hline
4x^2-14x-3 & \qquad -12x^2+42x+9 \\
 & \qquad(+)\;(-)\;(-) \\
 & \qquad -12x^2+42x+9 \\
\hline
 & \qquad\quad 0
\end{array}$$

$$\therefore \sqrt{4x^4-28x^3+37x^2+42x+9} = |\,2x^2-7x-3\,|$$

(iii) $\mathbf{16x^4 + 8x^2 + 1}$

Hence some terms are absent so let

$$16x^4 + 0x^3 + 8x^2 + 0x + 1$$

$$\begin{array}{r|l}
 & 4x^2 + 0x + 1 \\
\hline
4x^2 & 16x^{14} + 0x^3 + 8x^2 + 0x + 1 \\
 & (-) \\
 & 16x^4 \\
\hline
8x^2 + 0x + 1 & \quad 0x^3 + 8x^2 + 0x + 1 \\
 & \quad (-) \quad (-) \quad (-) \quad (-) \\
 & \quad 0x^3 + 8x^2 + 0x + 1 \\
\hline
 & \qquad 0
\end{array}$$

$\therefore \sqrt{16x^4 + 8x^2 + 1} = |\, 4x^2 + 1 \,|$

(iv) $\mathbf{121x^4 - 198x^3 - 183x^2 + 216x + 144}$

$$\begin{array}{r|l}
 & 11x^2 - 9x - 12 \\
\hline
11x^2 & 121x^4 - 198x^3 - 183x^2 + 216x + 144 \\
 & (-) \\
 & 121x^4 \\
\hline
22x^2 - 9x & \quad -198x^3 - 183x^2 \\
 & \quad (+) \quad (-) \\
 & \quad -198x^3 + 81x^2 \\
\hline
22x^2 - 18x - 12 & \qquad -264x^2 + 216x + 144 \\
 & \qquad (+) \quad (-) \quad (-) \\
 & \qquad -264x^2 + 216x + 144 \\
\hline
 & \qquad 0
\end{array}$$

$\therefore \sqrt{121x^4 - 198x^3 - 183x^2 + 216x + 144}$

$= |\, 11x^2 - 9x - 12 \,|$

Example 3.21

Find the square root of $\mathbf{64x^4 - 16x^3 + 17x^2 - 2x + 1}$

$$\begin{array}{r|l}
 & 8x^2 - x + 1 \\
\hline
8x^2 & 64x^4 - 16x^3 + 17x^2 - 2x + 1 \\
 & (-) \\
 & 64x^4 \\
\hline
16x^2 - x & \quad -16x^3 + 17x^2 \\
 & \quad (+) \quad (-) \\
16x^2 - 2x + 1 & \quad -16x^3 + x^2 \\
\hline
 & \quad 16x^2 - 2x + 1 \\
 & \quad (-) \quad (+) \quad (-) \\
 & \quad 16x^2 - 2x + 1 \\
\hline
 & \qquad 0
\end{array}$$

$\therefore \sqrt{64x^4 - 16x^3 + 17x^2 - 2x + 1} = |\, 8x^2 - x + 1 \,|$

Example 3.22

Find the square root of the expression $\mathbf{\frac{4x^2}{y^2} + \frac{20x}{y} + 13 - \frac{30y}{x} + \frac{9y^2}{x^2}}$

$$\begin{array}{r|l}
 & \frac{2x}{y} + 5 - \frac{3y}{x} \\
\hline
\frac{2x}{y} & \frac{4x^2}{y^2} + \frac{20x}{y} + 13 - \frac{30y}{x} + \frac{9y^2}{x^2} \\
 & (-) \\
 & \frac{4x^2}{y^2} \\
\hline
\frac{4x}{y} + 5 & \quad \frac{20x}{y} + 13 \\
 & \quad (-) \quad (-) \\
 & \quad \frac{20x}{y} + 25 \\
\hline
\frac{4x}{y} + 10 - \frac{3y}{x} & \qquad -12 - \frac{30y}{x} + \frac{9y^2}{x^2} \\
 & \qquad (+) \quad (+) \quad (-) \\
 & \qquad -12 - \frac{30y}{x} + \frac{9y^2}{x^2} \\
\hline
 & \qquad 0
\end{array}$$

$\therefore \sqrt{\frac{4x^2}{y^2} + \frac{20x}{y} + 13 - \frac{30y}{x} + \frac{9y^2}{x^2}} = \left|\, \frac{2x}{y} + 5 - \frac{3y}{x} \,\right|$

2. Find the square root of the expression $\mathbf{\frac{x^2}{y^2}-\frac{10x}{y}+27-\frac{10y}{x}+\frac{y^2}{x^2}}$

$$\begin{array}{r|l}
 & \frac{x}{y}-5+\frac{y}{x} \\ \hline
\frac{x}{y} & \frac{x^2}{y^2}-\frac{10x}{y}+27-\frac{10y}{x}+\frac{y^2}{x^2} \\
 & (-) \\
 & \frac{x^2}{y^2} \\ \hline
\frac{2x}{y}-5 & -\frac{10x}{y}+27 \\
 & (+)\quad(-) \\
 & -\frac{10x}{y}+25 \\ \hline
\frac{2x}{y}-10+\frac{y}{x} & 2-\frac{10y}{x}+\frac{y^2}{x^2} \\
 & (-)\quad(+)\quad(-) \\
 & 2-\frac{10y}{x}+\frac{y^2}{x^2} \\ \hline
 & 0
\end{array}$$

$$\sqrt{\frac{x^2}{y^2}-\frac{10x}{y}+27-\frac{10y}{x}+\frac{y^2}{x^2}} = \left|\frac{x}{y}-5+\frac{y}{x}\right|$$

Type II: Given expressions are perfect square find unknown values

Q.No. 3(i)(ii), Example 3.23, 4(i)(ii)

3. Find the values of a and b if the following polynomials are perfect squares

(i) $4x^4-12x^3+37x^2+bx+a$

$$\begin{array}{r|l}
 & 2x^2-3x+7 \\ \hline
2x^2 & 4x^4-12x^3+37x^2+bx+a \\
 & (-) \\
 & 4x^4 \\ \hline
4x^2-3x & -12x^3+37x^2 \\
 & (+)\quad(-) \\
 & -12x^3+9x^2 \\ \hline
4x^2-6x+7 & 28x^2+bx+a \\
 & (-)\quad(+)\quad(-) \\
 & 28x^2-42x+49 \\ \hline
 & 0
\end{array}$$

Because the given polynomial is a perfect square $a-49=0$; $b+42=0$

$a=49$ $b=-42$

(ii) $ax^4+bx^3+361x^2+220x+100$

$$\begin{array}{r|l}
 & 10+11x+12x^2 \\ \hline
10 & 100+220x+361x^2+bx^3+ax^4 \\
 & (-) \\
 & 100 \\ \hline
20+11x & 220x+361x^2 \\
 & (-)\quad(-) \\
 & 220x+121x^2 \\ \hline
20+22x+12x^2 & 240x^2+bx^3+ax^4 \\
 & 240x^2+264x^3+144x^4 \\ \hline
 & 0
\end{array}$$

$\therefore$ $a=144$

$b=264$

Example 3.23

If $9x^4+12x^3+28x^2+ax+b$ is a perfect square, find the values of a and b

$$\begin{array}{r|l}
 & 3x^2+2x+4 \\ \hline
3x^2 & 9x^4+12x^3+28x^2+ax+b \\
 & (-) \\
 & 9x^4 \\ \hline
6x^2+2x & 12x^3+28x^2 \\
 & (-)\quad(-) \\
 & 12x^3+4x^2 \\ \hline
6x^2+4x+4 & 24x^2+ax+b \\
 & (-)\quad(-)\quad(-) \\
 & 24x^2+16x+16 \\ \hline
 & 0
\end{array}$$

Because the given polynomial is a perfect square

$a-16=0$	$b-16=0$
$a=16$	$b=16$

4. Find the values of m and n if the following expressions are perfect squares.

(i) $\dfrac{1}{x^4}-\dfrac{6}{x^3}+\dfrac{13}{x^2}+\dfrac{m}{x}+n$

$$
\begin{array}{r|l}
 & \dfrac{1}{x^2}-\dfrac{3}{x}+2 \\ \hline
\dfrac{1}{x^2} & \dfrac{1}{x^4}-\dfrac{6}{x^3}+\dfrac{13}{x^2}+\dfrac{m}{x}+n \\
 & (-) \\
 & \dfrac{1}{x^4} \\ \hline
\dfrac{2}{x^2}-\dfrac{3}{x} & -\dfrac{6}{x^3}+\dfrac{13}{x^2} \\
 & (+) \quad (-) \\
 & -\dfrac{6}{x^3}+\dfrac{9}{x^2} \\ \hline
\dfrac{2}{x^2}-\dfrac{6}{x}+2 & \dfrac{4}{x^2}+\dfrac{m}{x}+n \\
 & \dfrac{4}{x^2}-\dfrac{12}{x}+4 \\ \hline
 & 0
\end{array}
$$

$\therefore \quad m=-12$

$n=4$

(ii) $x^4-8x^3+mx^2+nx+16$

$$
\begin{array}{r|l}
 & x^2-4x+4 \\ \hline
x^2 & x^4-8x^3+mx^2+nx+16 \\
 & (-) \\
 & x^4 \\ \hline
2x^2-4x & -8x^3+mx^2 \\
 & (+) \quad (-) \\
 & -8x^3+16x^2 \\ \hline
2x^2-8x+4 & (m-16)x^2+mx+16 \\
 & (-) \quad (+) \quad (-) \\
 & 8x^2-32x+16 \\ \hline
 & 0
\end{array}
$$

$m-16-8=0$

$m=16+8$

$\boxed{m=24}$

$n+32=0$

$\boxed{n=-32}$

Exercise 3.9

Key points

1. Quadratic Expression

An expressions of degree 2 is called a quadratic expression which is expressed as $P(x)=ax^2+bx+c,\ a\neq 0$ and a, b, c are real numbers.

2. Zeros of a quadratic polynomial

Let $P(x)$ be a polynomial $x=a$ is called zero of $P(x)$ if $P(a)=0$.

3. Roots of quadratic equations

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

4. Formation of a quadratic equation

$x^2-(\text{sum of roots})\,x+\text{Product of roots}=0$

i.e $x^2-(\alpha+\beta)\,x+\alpha\beta=0$

5. Sum and product of the roots of a quadratic equation

(i) Sum of roots $=\dfrac{-\text{co}-\text{eff of } x}{\text{co}\cdot\text{eff of } x^2}$

$$\boxed{\alpha\beta=\frac{-b}{a}}$$

(ii) Product of roots $=\dfrac{\text{constant}}{\text{co}\cdot\text{eff of } x^2}$

$$\boxed{\alpha\beta=\frac{c}{a}}$$

Type I: [Determine the quadratic equations: Q.No. 1(i) to (iv), Example 3.25(i) (ii) (iii)

1. Determine the quadratic equations whose sum and product of roots are

(i) $-9, 20$

Sum $(\alpha+\beta)=-9$

Product $(\alpha\beta)=20$

$\therefore$ **Equation**

$$x^2 - (\alpha + \beta)\, x + \alpha\,\beta = 0$$

$$x^2 - (-9)\, x + 20 = 0$$

$$x^2 + 9x + 20 = 0$$

(ii) $\mathbf{\frac{5}{3}, 4}$

Sum $(\alpha + \beta) = \frac{5}{3}$

Product$(\alpha\,\beta) = 4$

Equation

$$x^2 - (\alpha + \beta)\, x + \alpha\,\beta = 0$$

$$x^2 - \frac{5}{3}x + 4 = 0$$

$$3x^2 - 5x + 12 = 0$$

(iii) $\mathbf{\frac{-3}{2}, -1}$

Sum $(\alpha + \beta) = \frac{-3}{2}$

Product$(\alpha\,\beta) = -1$

Equation

$$x^2 - (\alpha + \beta)\, x + \alpha\,\beta = 0$$

$$x^2 - \left(\frac{-3}{2}\right) x + (-1) = 0$$

$$x^2 + \frac{3}{2}x - 1 = 0$$

$$2x^2 + 3x - 2 = 0$$

(iv) $\mathbf{-(2 - a)^2, (a + 5)^2}$

Sum $(\alpha + \beta) = -(2 - a)^2$

Product$(\alpha\,\beta) = (a + 5)^2$

Equation

$$x^2 - (\alpha + \beta)\, x + \alpha\,\beta = 0$$

$$x^2 + (2 - a)^2\, x + (a + 5)^2 = 0$$

Example 3.25

Write down the quadratic equation is general form for which sum and product of the roots are given below.

(i) 9, 14

Sum = 9

Product = 14

Equation

$$x^2 - (\text{sum})\, x + \text{product} = 0$$

$$x^2 - 9x + 14 = 0$$

(ii) $\mathbf{\frac{-7}{2}, \frac{5}{2}}$

Sum $(\alpha + \beta) = \frac{-7}{2}$

Product $(\alpha\,\beta) = \frac{5}{2}$

Equation

$$x^2 - (\text{sum})\, x + \text{product} = 0$$

$$x^2 + \frac{7}{2}x + \frac{5}{2} = 0$$

(multiply by 2)

$$2x^2 + 7x + 5 = 0$$

(iii) $\mathbf{\frac{-3}{5}, \frac{-1}{2}}$

Sum $(\alpha + \beta) = \frac{-3}{5}$

Product $(\alpha\,\beta) = \frac{-1}{2}$

Equation

$$x^2 - (\text{sum})\, x + \text{product} = 0$$

$$x^2 + \frac{3}{5}x - \frac{1}{2} = 0$$

(Multiply by 10)

$$10x^2 + 6x - 5 = 0$$

Type II: Find the sum and product of the quadratic equations

Q.No.2(i) (ii) (iii) (iv), Example 3.26(i) (ii) (iii), Example 3.24.

2. Find the sum and product of the roots for each of the following quadratic equation:

(i) $x^2 + 3x - 28 = 0$

$x^2 + 3x - 28 = 0$

$a = 1 \,;\, b = 3 \,;\, c = -28$

- Sum of roots $= \dfrac{-b}{a}$

$= \dfrac{-3}{1}$

$\alpha + \beta = -3$

- Product of roots $= \dfrac{c}{a}$

$= \dfrac{-28}{1}$

$\alpha\beta = -28$

(ii) $x^2 + 3x = 0$

$a = 1 \,;\, b = 3 \,;\, c = 0$

- Sum of roots $= \dfrac{-b}{a}$

$= \dfrac{-3}{1}$

$\alpha + \beta = -3$

- Product of roots $= \dfrac{c}{a}$

$= \dfrac{0}{1}$

$\alpha\beta = 0$

(iii) $3 + \dfrac{1}{a} = \dfrac{10}{a^2}$

$$3 + \frac{1}{a} = \frac{10}{a^2}$$

$$3 + \frac{1}{a} - \frac{10}{a^2} = 0$$

Multiply by a^2

$$3a^2 + a - 10 = 0$$

Here $A = 3 \,;\, B = 1 \,;\, C = -10$

- Sum of roots $= \dfrac{-B}{A}$

$\alpha + \beta = \dfrac{-1}{3}$

- Product of roots $= \dfrac{C}{A}$

$\alpha\beta = \dfrac{-10}{3}$

(iv) $3y^2 - y - 4 = 0$

$a = 3 \,;\, b = -1 \,;\, c = -4$

- Sum of roots $= \dfrac{-b}{a}$

$\alpha + \beta = \dfrac{1}{3}$

- Product of roots $= \dfrac{c}{a}$

$\alpha\beta = \dfrac{-4}{3}$

Example 3.26

Find the sum and product of the roots for each of the following quadratic equations.

(i) **$x^2 + 8x - 65 = 0$**

$a = 1 \,;\, b = 8 \,;\, c = -65$

- Sum of roots $= \dfrac{-b}{a}$

$= -8$

- Product of roots $= \dfrac{c}{a}$

$= -65$

(ii) $2x^2 + 5x + 7 = 0$

$a = 2 \,;\, b = 5 \,;\, c = 7$

- Sum of roots $= \frac{-b}{a}$

$= \frac{-5}{2}$

- Product of roots $= \frac{c}{a}$

$= \frac{7}{2}$

(iii) $\boldsymbol{kx^2 - k^2 x - 2k^3 = 0}$

$a = k\ ;\ b = -k^2\ ;\ c = -2k^3$

- Sum of roots $= \frac{-b}{a}$

$= \frac{k^2}{k}$

$= k$

- Product of roots $= \frac{c}{a}$

$= \frac{-2k^3}{k}$

$= -2k^2$

Example 3.24

Find the zeroes of the quadratic expression $\boldsymbol{x^2 + 8x + 12}$

Let $P(x) = x^2 + 8x + 12$

$= (x+2)(x+6)$

$P(-2) = (-2)^2 + 8(-2) + 12$

$= 4 - 16 + 12$

$= 0$

$P(-6) = (-6)^2 + 8(-6) + 12$

$= 36 - 48 + 12$

$= 0$

$\therefore$ -2 and -6 are zeroes of $P(x)$.

Exercise 3.10

Key points

Solving quadratic equations

1. Factorization method
2. Quadratic formula method

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Completing the square method

1. Solve the following quadratic equations by factorization method

(i) $\boldsymbol{4x^2 - 7x - 2 = 0}$

$4x^2 - 7x - 2 = 0$

$(x-2)(4x+1) = 0$

$x - 2 = 0$ | $4x + 1 = 0$

$x = 2$ | $x = \frac{-1}{4}$

$\therefore$ roots are $x = 2, \frac{-1}{4}$

-8: -8, 1; $\frac{-8}{4}$, $\frac{1}{4}$; -2, $\frac{1}{4}$

(ii) $\boldsymbol{3(P^2 - 6) = P(P+5)}$

$3P^2 - 18 = P^2 + 5P$

$3P^2 - 18 - P^2 - 5P = 0$

$2P^2 - 5P - 18 = 0$

$(2P - 9)(P + 2) = 0$

$2P - 9 = 0$ | $P + 2 = 0$

$2P = 9$ | $P = -2$

$P = \frac{9}{2}$,

$\therefore$ roots are -2 and $\frac{9}{2}$

-36: -9, 4; $\frac{-9}{2}$, $\frac{4}{2}$; $\frac{-9}{2}$, 2

(iii) $\boldsymbol{\sqrt{a(a-7)} = 3\sqrt{2}}$

$\sqrt{a(a-7)} = 3\sqrt{2}$

Squaring on both sides

$a(a-7) = (3\sqrt{2})^2$

$$a^2 - 7a = 18$$
$$a^2 - 7a - 18 = 0$$
$$(a-9)(a+2) = 0$$

-18

$-9 \quad 2$

$$a - 9 = 0 \quad\Big|\quad a + 2 = 0$$
$$a = 9 \quad\Big|\quad a = -2$$

$\therefore$ roots are $a = -2, 9$

(iv) $\mathbf{\sqrt{2}\,x^2 + 7x + 5\sqrt{2} = 0}$

$$\sqrt{2}\,x^2 + 7x + 5\sqrt{2} = 0$$
$$(x + \sqrt{2})(\sqrt{2}\,x + 5) = 0$$

$$x + \sqrt{2} = 0 \quad\Big|\quad \sqrt{2}\,x + 5 = 0$$
$$x = -\sqrt{2} \quad\Big|\quad \sqrt{2}\,x = -5$$
$$x = \frac{-5}{\sqrt{2}}$$

10

$2 \quad 5$

$\frac{2}{\sqrt{2}} \quad \frac{5}{\sqrt{2}}$

$\sqrt{2} \quad \frac{5}{\sqrt{2}}$

$\therefore$ roots are $x = -\sqrt{2}, \frac{-5}{\sqrt{2}}$

(v) $\mathbf{2x^2 - x + \frac{1}{8} = 0}$

Multiply by 8

$$16x^2 - 8x + 1 = 0$$
$$(4x-1)^2 = 0$$
$$4x - 1 = 0$$
$$4x = 1$$
$$x = \frac{1}{4}$$

16

$-4 \quad -4$

$\frac{-4}{16} \quad \frac{-4}{16}$

$\frac{-1}{4} \quad \frac{-1}{4}$

$\therefore$ roots are $\frac{1}{4}, \frac{1}{4}$

Example 3.27

Solve $\mathbf{2x^2 - 2\sqrt{6}\,x + 3 = 0}$

$$2x^2 - 2\sqrt{6}\,x + 3 = 0$$
$$2x^2 - \sqrt{6}\,x - \sqrt{6}\,x + 3 = 0$$
$$\sqrt{2}\,x(\sqrt{2}\,x - \sqrt{3}) - \sqrt{3}(\sqrt{2}\,x - \sqrt{3}) = 0$$
$$(\sqrt{2}\,x - \sqrt{3})(\sqrt{2}\,x - \sqrt{3}) = 0$$
$$\sqrt{2}\,x - \sqrt{3} = 0$$
$$\sqrt{2}\,x = \sqrt{3}$$
$$x = \frac{\sqrt{3}}{\sqrt{2}}$$

$\therefore$ roots are $\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}$

Example 3.28

Solve $\mathbf{2m^2 + 19m + 30 = 0}$

$$2m^2 + 19m + 30 = 0$$
$$(2m + 15)(m + 2) = 0$$

$$2m + 15 = 0 \quad\Big|\quad m + 2 = 0$$
$$2m = -15 \quad\Big|\quad m = -2$$
$$m = \frac{-15}{2}$$

60

$15 \quad 4$

$\frac{15}{2} \quad \frac{4}{2}$

$\frac{15}{2} \quad 2$

$\therefore$ roots are $-2, \frac{-15}{2}$

Example 3.29

Solve $\mathbf{x^4 - 13x^2 + 42 = 0}$

$$x^4 - 13x^2 + 42 = 0$$
$$(x^2)^2 - 13x^2 + 42 = 0$$

42

$-6 \quad -7$

Let $\quad a = x^2$

$$a^2 - 13a + 42 = 0$$
$$(a-6)(a-7) = 0$$

$$a - 6 = 0 \quad\Big|\quad a - 7 = 0$$
$$a = 6 \quad\Big|\quad a = 7$$
$$x^2 = 6 \quad\Big|\quad x^2 = 7$$
$$x = \pm\sqrt{6} \quad\Big|\quad x = \pm\sqrt{7}$$

$\therefore$ roots are $x = \pm\sqrt{6}, \pm\sqrt{7}$

Example 3.30

Solve $\mathbf{\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}}$

$$\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}$$

Let $y = \frac{x}{x-1}$ then $\frac{1}{y} = \frac{x-1}{x}$

$$\therefore \quad y + \frac{1}{y} = 2\frac{1}{2}$$

$$\frac{y^2+1}{y} = \frac{5}{2}$$

$$2y^2 + 2 = 5y$$

$$2y^2 - 5y + 2 = 0$$

$$(y-2)\,(2y-1) = 0$$

$$\begin{array}{c} 4 \\ -4 \quad -1 \\ \frac{-4}{2} \quad \frac{-1}{2} \\ -2 \quad \frac{-1}{2} \end{array}$$

$y - 2 = 0$	$2y - 1 = 0$
$y = 2$	$2y = 1$
$\frac{x}{x-1} = 2$	$y = \frac{1}{2}$
$x = 2x - 2$	$\frac{x}{x-1} = \frac{1}{2}$
$x = 2$	$2x = x - 1$
	$x = -1$

$\therefore$ roots are $x = -1, 2$

---

2. The numbers of volleyball games that must be scheduled in a league with 'n' teams is given by $G(n) = \frac{n^2 - n}{2}$ where each team plays with every other team exactly once. A league schedules 15 games. How many teams are in the league?

Given

$$G(n) = \frac{n^2 - n}{2}$$

$$15 = \frac{n^2 - n}{2}$$

$$n^2 - n = 30$$

$$n^2 - n - 30 = 0$$

$$(n-6)\,(n+5) = 0$$

$$\begin{array}{c} -30 \\ -6 \quad 5 \end{array}$$

$n - 6 = 0$	$n + 5 = 0$
$n = 6$	$n = -5$ (not possible)

$\therefore$ Number of teams in the league is 6.

Exercise 3.11

KEY POINTS

II. Solving a Quadratic equation by completing the square method.

Step 1: Write the quadratic equation in general form $ax^2 + bx + c = 0$

Step 2: Divide both sides of the equation by the co-efficient of x^2 if it is not 1.

Step 3: Shift the constant term to the right hand side.

Step 4: Add the square of one-half of the co-efficient of x to both sides.

Step 5: Write the left hand sides as a square and simplify the right hand side.

Step 6: Take the square root on both sides and solve for x.

III. Solving a quadratic equation by formula method.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Type I: Completing the square method Q.No. 1(i)(ii), Example 3.31, 3.32

1. Solve the following quadratic equations by completing the square method:

(i) $9x^2 - 12x + 4 = 0$

$9x^2 - 12x + 4 = 0$

Divide by 9

$$x^2 - \frac{12}{9}x + \frac{4}{9} = 0$$

$$x^2 - \frac{4}{3}x = \frac{-4}{9}$$

$$\left[\frac{1}{2}\text{ co. eff } x = \frac{4}{3} \times \frac{1}{2} = \frac{2}{3}\right]$$

Add $\left(\frac{2}{3}\right)^2$ on both sides

$$x^2 - \frac{4}{3}x + \left(\frac{2}{3}\right)^2 = \frac{-4}{9} + \frac{4}{9}$$

$$\left(x-\frac{2}{3}\right)^2=0$$

$$x-\frac{2}{3}=0$$

$$x=\frac{2}{3}$$

$\therefore$ The roots are $x=\frac{2}{3},\frac{2}{3}$

(ii) $\mathbf{\dfrac{5x+7}{x-1}=3x+2}$

$$\frac{5x+7}{x-1}=3x+2$$

$$5x+7=(3x+2)(x-1)$$

$$5x+7=3x^2-3x+2x-2$$

$$5x+7=3x^2-x-2$$

$$3x^2-x-2-5x-7=0$$

$$3x^2-6x-9=0$$

Divided by 3

$$x^2-2x-3=0$$

$$x^2-2x=3$$

[Add $(1)^2$ on both sides]

$$x^2-2x+1^2=3+1^2$$

$$(x-1)^2=4$$

$$(x-1)^2=2^2$$

$$(x-1)^2-2^2=0$$

[use $a^2-b^2=(a+b)(a-b)$

$$(x-1+2)(x-1-2)=0$$

$$(x+1)(x-3)=0$$

$x+1=0$ $x-3=0$

$x=-1$ $x=3$

$\therefore$ roots are $x=-1, 3$

Example 3.31

Solve $\mathbf{x^2-3x-2=0}$

$$x^2-3x-2=0$$

$$x^2-3x=2$$

Add $\left(\frac{3}{2}\right)^2$ on both sides

$$x^2-3x+\left(\frac{3}{2}\right)^2=2+\left(\frac{3}{2}\right)^2$$

$$\left(x-\frac{3}{2}\right)^2=2+\frac{9}{4}$$

$$\left(x-\frac{3}{2}\right)^2=\frac{17}{4}$$

$$x-\frac{3}{2}=\pm\sqrt{\frac{17}{4}}$$

$x-\frac{3}{2}=\frac{\sqrt{17}}{2}$	$x-\frac{3}{2}=\frac{-\sqrt{17}}{2}$
$x=\frac{\sqrt{17}}{2}+\frac{3}{2}$	$x=\frac{-\sqrt{17}}{2}+\frac{3}{2}$
$x=\frac{3+\sqrt{17}}{2}$	$x=\frac{3-\sqrt{17}}{2}$

$\therefore$ Roots are $x=\frac{3+\sqrt{17}}{2},\frac{3-\sqrt{17}}{2}$

Example 3.32

Solve $\mathbf{2x^2-x-1=0}$

$$2x^2-x-1=0$$

divide by 2

$$x^2-\frac{x}{2}-\frac{1}{2}=0$$

$$x^2-\frac{x}{2}=\frac{1}{2}$$

Add $\left(\frac{1}{4}\right)^2$ on both sides

$$x^2-\frac{x}{2}+\left(\frac{1}{4}\right)^2=\frac{1}{2}+\left(\frac{1}{4}\right)^2$$

$$\left(x-\frac{1}{4}\right)^2=\frac{1}{2}+\frac{1}{16}$$

$$\left(x-\frac{1}{4}\right)^2=\frac{8+1}{16}$$

$$\left(x-\frac{1}{4}\right)^2=\frac{9}{16}$$

$$x-\frac{1}{4}=\pm\frac{3}{4}$$

$x-\frac{1}{4}=\frac{3}{4}$	$x-\frac{1}{4}=\frac{-3}{4}$
$x=\frac{3}{4}+\frac{1}{4}$	$x-\frac{1}{4}=\frac{-3}{4}$
$x=\frac{4}{4}$	$x=\frac{-3}{4}+\frac{1}{4}$
$x=1$	$x=\frac{-2}{4}$
	$x=\frac{-1}{2}$

$\therefore$ roots are $\frac{-1}{2}, 1$

Type II: Quadratic formula method

Q.No. 2(i) (ii) (iii), Example 3.33, 3.34, 3.35, 3.36, 2(iv), 3

2. Solve the following quadratic equations by formula method:

(i) $2x^2-5x+2=0$

$2x^2-5x+2=0$

$a=2\,;\,b=-5\,;\,c=2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{5\pm\sqrt{25-4\,(2)(2)}}{2\,(2)}$$

$$=\frac{5\pm\sqrt{25-16}}{4}$$

$$=\frac{5\pm\sqrt{9}}{4}$$

$$=\frac{5\pm3}{4}$$

$$=\frac{5+3}{4},\frac{5-3}{4}$$

$$x=\frac{8}{4},\frac{2}{4}\rightarrow x=2,\frac{1}{2}$$

(ii) $\sqrt{2}\,f^2-6f+3\sqrt{2}=0$

$a=\sqrt{2}\,;\,b=-6\,;\,c=3\sqrt{2}$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{6\pm\sqrt{36-4\,(\sqrt{2})\,(3\sqrt{2})}}{2\sqrt{2}}$$

$$=\frac{6\pm\sqrt{36-24}}{2\sqrt{2}}$$

$$=\frac{6\pm\sqrt{12}}{2\sqrt{2}}$$

$$=\frac{6\pm2\sqrt{3}}{2\sqrt{2}}$$

$$=\frac{2\,(3\pm\sqrt{3})}{2\sqrt{2}}$$

$$x=\frac{3+\sqrt{3}}{\sqrt{2}},\frac{3-\sqrt{3}}{\sqrt{2}}$$

(iii) $3y^2-20y-23=0$

$a=3\,;\,b=-20\,;\,c=-23$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{20\pm\sqrt{400-4\,(3)\,(-23)}}{2\,(3)}$$

$$=\frac{20\pm\sqrt{400+276}}{6}$$

$$=\frac{20\pm\sqrt{676}}{6}$$

$$=\frac{20\pm26}{6}$$

$$=\frac{20+26}{6},\frac{20-26}{6}$$

$$=\frac{46}{6},\frac{-6}{6}\Rightarrow x=\frac{23}{3},-1$$

Example 3.33

Solve $x^2 + 2x - 2 = 0$ by formula method.

$$x^2 + 2x - 2 = 0$$

$a = 1, b = 2, c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 4\,(1)\,(-2)}}{2\,(1)}$$

$$= \frac{-2 \pm \sqrt{4 + 8}}{2}$$

$$= \frac{-2 \pm \sqrt{12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}}{2}$$

$$= \frac{2\,(-1 \pm \sqrt{3})}{2}$$

$x = -1 + \sqrt{3}\ ;\ -1 - \sqrt{3}$

Example 3.34

Solve $2x^2 - 3x - 3 = 0$ by formula method

$2x^2 - 3x - 3 = 0$

$a = 2\ ;\ b = -3\ ;\ c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{9 - 4\,(2)\,(-3)}}{2\,(2)}$$

$$= \frac{3 \pm \sqrt{9 + 24}}{4}$$

$$= \frac{3 \pm \sqrt{33}}{4}$$

$$x = \frac{3 + \sqrt{33}}{4}, \frac{3 - \sqrt{33}}{4}$$

Example 3.35

Solve $3P^2 + 2\sqrt{5}\,P - 5 = 0$ by formula method.

$$3P^2 + 2\sqrt{5}\,P - 5 = 0$$

$a = 3\ ;\ b = 2\sqrt{5}\ ;\ c = -5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2\sqrt{5} \pm \sqrt{(2\sqrt{5})^2 - 4\,(3)\,(-5)}}{2\,(3)}$$

$$= \frac{-2\sqrt{5} \pm \sqrt{20 + 60}}{6}$$

$$= \frac{-2\sqrt{5} \pm \sqrt{80}}{6}$$

$$= \frac{-2\sqrt{5} \pm 4\sqrt{5}}{6}$$

$$= \frac{2\,(-\sqrt{5} \pm 2\sqrt{5})}{6}$$

$$= \frac{-\sqrt{5} \pm 2\sqrt{5}}{3}$$

$$x = \frac{-\sqrt{5} + 2\sqrt{5}}{3}, \frac{-\sqrt{5} - 2\sqrt{5}}{3}$$

$$= \frac{\sqrt{5}}{3}, \frac{-3\sqrt{5}}{3}$$

$$x = \frac{\sqrt{5}}{3}, -\sqrt{5}$$

Example 3.36

Solve $pq\,x^2 - (p + q)^2\,x + (p + q)^2 = 0$

Here $a = pq$

$b = -(p + q)^2$

$c = (p + q)^2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{(p+q)^2 \pm \sqrt{\left[-(p+q)^2\right]^2 - 4pq\,(p+q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^4 - 4pq\,(p+q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2\,[(p+q)^2 - 4pq]}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2\,[p^2 + 2pq + q^2 - 4pq]}}{2pq}$$

$$= \frac{(p+q)^2 \pm \sqrt{(p+q)^2\,(p-q)^2}}{2pq}$$

$$= \frac{(p+q)^2 \pm (p+q)\,(p-q)}{2pq}$$

$$= \frac{(p+q)\,[(p+q) \pm (p-q)]}{2pq}$$

$$x = \frac{(p+q)\,[(p+q)+(p-q)]}{2pq}$$

$$= \frac{(p+q)\,(2p)}{2pq}$$

$$= \frac{p+q}{q}$$

$$= \frac{(p+q)\,[(p+q)-(p-q)]}{2pq}$$

$$= \frac{(p+q)\,[p+q-p+q]}{2pq}$$

$$= \frac{(p+q)\,(2q)}{2pq}$$

$$= \frac{p+q}{p}$$

✍ **Solution:**

$$x = \frac{p+q}{q}, \frac{p+q}{p}$$

2. (iv) $36y^2 - 12ay + (a^2 - b^2) = 0$

$A = 36$; $B = -12a$; $C = a^2 - b^2$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$x = \frac{12a \pm \sqrt{(-12a)^2 - 4\,(36)\,(a^2 - b^2)}}{2\,(36)}$$

$$= \frac{12a \pm \sqrt{144a^2 - 144a^2 + 144b^2}}{72}$$

$$= \frac{12a \pm \sqrt{144\,b^2}}{72}$$

$$= \frac{12a \pm 12b}{72}$$

$$= \frac{12\,(a \pm b)}{72}$$

$$= \frac{a \pm b}{6}$$

$$x = \frac{a+b}{6} ; \frac{a-b}{6}$$

---

3. A ball rolls down a slope and travels a distance $d = t^2 - 0.75\,t$ feet in t seconds. Find the time when the distance travelled by the ball is 11.25 feet.

Given

$$d = t^2 - 0.75\,t$$

$$11.25 = t^2 - 0.75\,t$$

$$t^2 - 0.75\,t - 11.25 = 0$$

$a = 1$; $b = -0.75$; $c = -11.25$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{0.75 \pm \sqrt{0.5625 - 4\,(-11.25)}}{2}$$

$$= \frac{0.75 \pm \sqrt{0.5625 + 45}}{2}$$

$$= \frac{0.75 \pm \sqrt{45.5625}}{2}$$

$$= \frac{0.75 \pm 6.75}{2}$$

	6.75
6	45.56,25
	36
127	956
	889
1345	6725
	6725
	0

$$t = \frac{0.75 + 6.75}{2}, \frac{0.75 - 6.75}{2}$$

$$t = \frac{7.50}{2}, \frac{-6}{2}$$

$$= 3.75, -3$$

Negative not possible.

$\therefore$ Time taken $t = 3.75$ seconds

Exercise 3.12

Key points

Solving problems involving quadratic equations

Step 1: Convert the word problem to a quadratic equation form.

Step 2: Solve the quadratic equation obtained in any one of the above three methods.

Step 3: Relate the mathematical solution obtained to the statement asked in the question.

Type I: Simple problems
Q.No. 1, 6, Example 3.39

1. If the difference between a number and its reciprocal is $\frac{24}{5}$, find the number.

Let the number be x

Given $x - \frac{1}{x} = \frac{24}{5}$

$$\frac{x^2 - 1}{x} = \frac{24}{5}$$

$$5x^2 - 5 = 24x$$

$$5x^2 - 24x - 5 = 0$$

$$(x - 5)(5x + 1) = 0$$

-25

$-25 \quad 1$

$\frac{-25}{5} \quad \frac{1}{5}$

$-5 \quad \frac{1}{5}$

$x - 5 = 0$	$5x + 1 = 0$
$x = 5$	$5x = -1$
	$x = \frac{-1}{5}$

$\therefore$ The number is $5, \frac{-1}{5}$

---

6. From a group of $2x^2$ black bees, square root of half of the group went to a tree. Again eight, ninth of the bees went to the same tree. The remaining two got caught up in a fragrant lotus. How many bees were there in total?

Given

Number of black bees $= 2x^2$

Given $2x^2 - x - \frac{8}{9}(2x^2) = 2$

$$18x^2 - 9x - 16x^2 = 18$$

$$2x^2 - 9x - 18 = 0$$

$$(x - 6)(2x + 3) = 0$$

-36

$-12 \quad 3$

$\frac{-12}{2} \quad \frac{3}{2}$

$-6 \quad \frac{3}{2}$

$x - 6 = 0$	$2x + 3 = 0$
$x = 6$	$x = \frac{-3}{2}$ (not possible)

$\therefore$ Number of bees $= 2x^2$

$= 2(6)^2$

$= 2 \times 36$

$= 72$

Example 3.39

A flock of swans contained x^2 members. As the clouds gathered, $10x$ went to a lake and one-eight of the members flew away to a garden. The remaining three pairs played about in the water. How many swans were there in total?

As given there are x^2 swans.

Given data, $x^2 - 10x - \frac{1}{8}x^2 = 6$

$$8x^2 - 80x - x^2 = 48$$

$$7x^2 - 80x - 48 = 0$$

$a = 7 ; b = -80 ; c = -48$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{80 \pm \sqrt{6400 - 4(7)(-48)}}{2(7)}$$

$$= \frac{80 \pm \sqrt{6400 + 1344}}{14}$$

$$= \frac{80 \pm \sqrt{7744}}{14}$$

	88
8	77,44
	64
168	1344
	1344
	0

$= \frac{80 \pm 88}{14}$

$= \frac{80+88}{14}, \quad \frac{80-88}{14}$

$= \frac{168}{14}, \quad \frac{-8}{14}$

$= 12, \quad \frac{-4}{7}$ Not possible

$\therefore$ Number of swans $= (12)^2 = 144$.

Type II: Comparison based sums
Q.No. 4, Example 3.37, 9

4. A girl is twice as old as her sister. Five years hence, the product of their ages (in years) will be 375. Find their present ages.

Let the present age of the girl is '$2x$' years and her sister age is x years.

After five years

Given

$$(2x+5)(x+5) = 375$$

$$2x^2 + 10x + 5x + 25 - 375 = 0$$

$$2x^2 + 15x - 350 = 0$$

$$(x-10)(2x+35) = 0$$

$x - 10 = 0$	$2x + 35 = 0$
$x = 10$	$x = \frac{-35}{2}$
	not posible

$\therefore$ Age of sister = 10 years

Age of girl $= 2(10)$

$= 20$ years

-700

35	-20
$\frac{35}{2}$	$\frac{-20}{2}$
$\frac{35}{2}$	-10

Example 3.37

The product of Kumaran's age (in years) two years ago and his age fours years from now is one more than twice his present age. What is his present age?

Let the present age of Kumaran be x years.

Two years ago, his age $= (x-2)$ years fours years from now,

his age $= (x+4)$ years

Given $(x-2)(x+4) = 1 + 2x$

$$x^2 + 4x - 2x - 8 - 1 - 2x = 0$$

$$x^2 - 9 = 0$$

$$(x+3)(x-3) = 0$$

$x + 3 = 0$	$x - 3 = 0$
$x = -3$	$\boldsymbol{x = 3}$
not possible	

$\therefore$ present age of Kumaran is 3 years.

9. Two women together took 100 eggs to a market, one had more than the other. Both sold them for the same sum of money. The first then said to the second: "If I had your eggs, I would have earned Rs. 15", to which the second replied: "If I had your eggs, I would have earned Rs. $6\frac{2}{3}$". How many eggs did each had in the beginning?

Let the numbers of eggs of women 1 be x and woman 2 is y, and their selling price be p, q respectively.

Given $x + y = 100$ ...(1)

$$y = 100 - x$$

If both of them sold the eggs for the equal sum of money

$$px = qy \qquad ...(2)$$

Given

$py = 15$	$qx = 6\frac{2}{3}$
$py = 15$	$qx = \frac{20}{3}$
$p = \frac{15}{y}$	$q = \frac{20}{3x}$

Given $px = qy$

$$\frac{15}{y}x = \frac{20}{3x}y$$

$$\frac{15x}{100-x} = \frac{20\,(100-x)}{3x}$$

$$45x^2 = 20\,(100-x)^2$$

$$9x^2 = 4\,(x^2 - 200x + 10000)$$

$$9x^2 = 4x^2 - 800x + 40000$$

$$5x^2 + 800x - 40000 = 0$$

$$x^2 + 160x - 8000 = 0$$

$$(x+200)\,(x-40) = 0$$

-8000

$200 \quad -40$

$x + 200 = 0$	$x - 40 = 0$
$x = -200$	$x = 40$
not possible	

$$(1) \Rightarrow y = 100 - x$$

$$= 100 - 40$$

$$\boxed{y = 60}$$

$\therefore$ woman 1 had 40 eggs and woman 2 had 60 eggs.

Type III: Phythagoras theorem based sums:

Q.No. 10, Example 3.38, 5

10. The hypotenuse of a right angled triangle is 25 cm and its perimeter 56 cm. Find the length of the smallest side.

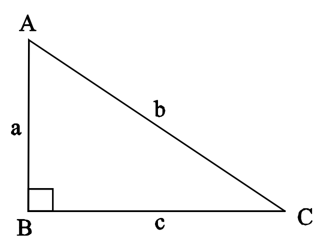
Given $b = 25$ cm

$$a + b + c = 56 \text{ cm}$$

$$a + c + 25 = 56$$

$$a + c = 56 - 25$$

$$a + c = 31$$



Let $a = x$

$\therefore \quad c = 31 - a$

$c = 31 - x$

In $\Delta\,ABC$, by phythogaras theorem

$$b^2 = a^2 + c^2$$

$$(25)^2 = (x)^2 + (31-x)^2$$

$$625 = x^2 + 961 - 62x + x^2$$

$$2x^2 - 62x + 961 - 625 = 0$$

$$2x^2 - 62x + 336 = 0$$

$$x^2 - 31x + 168 = 0$$

$$(x-24)\,(x-7) = 0$$

$x - 24 = 0$	$x - 7 = 0$
$x = 24$	$x = 7$

$\therefore$ The sides of triangle are 7 cm, 24 cm and 25 cm

Here smallest side = 7 cm.

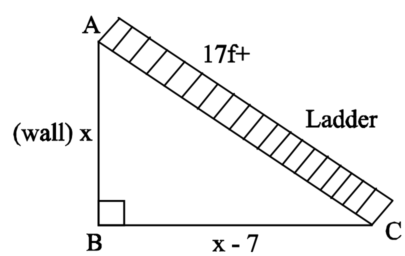
Example 3.38

A ladder 17 feet long is leaning against a wall. If the ladder, vertical wall and the floor from the bottom of the wall to the ladder form a right triangle, find the height of the wall where the top of the ladder meets if the distance between bottom of the wall to bottom of the ladder is 7 feet less than the height of the wall?

Let height of the wall AB = 'x' feet

Given data $BC = (x-7)$ feet

$AC = 17$ ft



by phythogaras theorem

$$AC^2 = AB^2 + BC^2$$

$$(17)^2 = x^2 + (x-7)^2$$

$$289 = x^2 + x^2 - 14x + 49$$

$$2x^2 - 14x + 49 - 289 = 0$$

$$2x^2 - 14x - 240 = 0$$

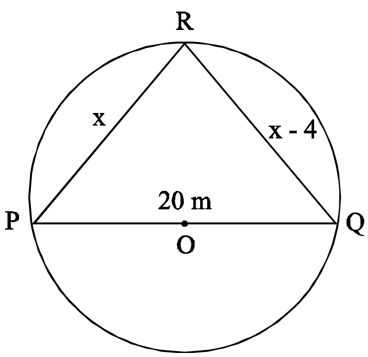
$$x^2 - 7x - 120 = 0$$
$$(x - 15)(x + 8) = 0$$

$x - 15 = 0$	$x + 8 = 0$
$x = 15$	$x = -8$ not possible.

$\therefore$ Height of the wall = 15 ft.

---

5. A pole has to be erected at a point on the boundary of a circular ground of diameter 20 m in such a way that the difference of its distance from two diametrically opposite fixed gates *P* and *Q* on the boundary is 4m. Is it possible to do so? If answer is yes at what distance from the two gates should the pole be erected?



$PQ = 20$ m is a diameter of a circle.
$\therefore \angle PRQ = 90°$ (Angle in a semi-circle)

Given data $PR = x$ then $QR = x - 4$

$\therefore$ In ΔPRQ

$$PQ^2 = PR^2 + QR^2$$
$$20^2 = x^2 + (x - 4)^2$$
$$400 = x^2 + x^2 - 8x + 16$$
$$2x^2 - 8x + 16 - 400 = 0$$
$$2x^2 - 8x - 384 = 0$$
$$x^2 - 4x - 192 = 0$$
$$(x - 16)(x + 12) = 0$$

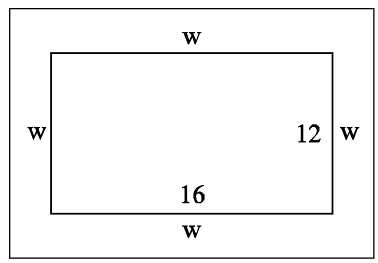
$x - 16 = 0$	$x + 12 = 0$
$x = 16$	$x = -12$ not possible.

$\therefore \quad QR = x - 4$
$= 16 - 4$
$= 12$ cm

$\therefore$ The pole should be erected at a distance of 16m and 12m from the two gates.

Type IV: Mensuration based sums
Q.No. 2, 8

2. A garden measuring 12m by 16m is to have a pedestrian pathway that is '*w*' meters wide installed all the way around so that it increases the total area to 285 m². What is the width of the pathway.



Dimension of outer rectangle

$$L = 16 + 2W$$
$$B = 12 + 2W$$

Area of outer rectangle $= 285 \text{ m}^2$

$$(16 + 2W)(12 + 2W) = 285$$
$$2(8 + W)\,2(6 + W) = 285$$
$$4[48 + 8W + 6W + W^2] = 285$$
$$4(W^2 + 14W + 48) - 285 = 0$$
$$4W^2 + 56W + 192 - 285 = 0$$
$$4W^2 - 56W - 93 = 0$$
$$(2W + 31)(2W - 3) = 0$$

$2W + 31 = 0$	$2W - 3 = 0$
$2W = -31$	$2W = 3$
$W = \frac{-31}{2}$	$W = \frac{3}{2}$

not possible

$\therefore$ Width of the pathway is 1.5m

2	372
2	186
3	93
	31

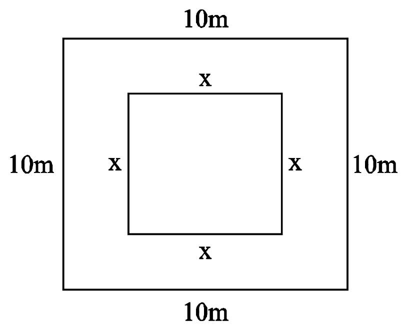
-372

$62 \quad -6$

$\frac{62}{4} \quad \frac{-6}{4}$

$\frac{31}{2} \quad \frac{-3}{2}$

8. There is a square field whose side is 10m. A square flower bed is preparallel in its centre leaving a gravel path all round the flowed bed. The total cost of laying the flower bed and gravelling the path at Rs. 3 and Rs. 4 per square metre respectively is Rs. 364. Find the width of the gravel path.



Outer square	**Inner square**
Side = 10 m	Side = 'x' m
Area $= 10 \times 10$	Area $= x^2$ m^2
$= 100$ m^2	

Area of the gravel path $= (100 - x^2)$ m^2

Given

Cost of laying flower bed $=$ Rs. 3 per m^2

Cost of laying gravel path $=$ Rs. 4 per m^2

Given data

$$3x^2 + 4\,(100 - x^2) = 364$$
$$3x^2 + 400 - 4x^2 - 364 = 0$$
$$-x^2 + 36 = 0$$
$$x^2 = 36$$
$$x = \pm 6$$

$\therefore$ Length of the flower bed = 6m

$\therefore$ Width of the path $\Rightarrow 6 + 2W = 10$

$$2W = 10 - 6$$
$$2W = 4$$
$$\boxed{W = 2}$$

$\therefore$ Width of the gravel path is 2m.

Type V: Distance speed, time based sums:
Q.No. 3, Example 3.40, 7

3. A bus covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hour more it would have taken 30 minutes less for the journey. Find the original speed of the bus.

Let the original speed of the bus = 'x' km/hr

New speed $= (x + 15)$ km/hr

distance $= 90$ km

Original

Time taken $(T_1) = \dfrac{\text{distance}}{\text{speed}}$

$$= \frac{90}{x} \text{ hrs}$$

New

Time taken $(T_2) = \dfrac{90}{x+15}$ hrs

$$\therefore \quad \frac{90}{x} - \frac{90}{x+15} = 30 \text{ minutes}$$
$$90\left(\frac{1}{x} - \frac{1}{x+15}\right) = \frac{1}{2}$$
$$90\left(\frac{x + 15 - x}{x\,(x+15)}\right) = \frac{1}{2}$$
$$\frac{1350}{x^2 + 15x} = \frac{1}{2}$$
$$x^2 + 15x = 2700$$
$$x^2 + 15x - 2700 = 0$$
$$(x + 60)\,(x - 45) = 0$$

5	2700
5	540
2	108
2	54
3	27
3	9
	3

-2700

$60 \quad -45$

$x + 60 = 0$	$x - 45 = 0$
$x = -60$	$\boldsymbol{x = 45}$
not possible	

$\therefore$ Original speed of the bus is 45 km/hr.

Example 3.40

A passenger train takes 1 hr more than an express train to travel a distance of 240 km from Chennai to Virudhachalam. The speed of passenger train is less than that of an express train by 20km per hour. Find the average speed of both the trains.

Let

Average speed of passenger train $= x$ km/hr

Average speed of express train

$= (x + 20)$ km/hr

Distance = 240 km.

$\therefore$ Time taken by the passenger train $= \dfrac{240}{x}$ hrs

Time taken by the express train $= \dfrac{240}{x+20}$ hr

Given

$$\frac{240}{x} - \frac{240}{x+20} = 1 \text{ hrs}$$

$$240\left(\frac{1}{x} - \frac{1}{x+20}\right) = 1$$

$$240\left(\frac{x+20-x}{x(x+20)}\right) = 1$$

$$\frac{4800}{x^2+20x} = 1$$

$$x^2 + 20x = 4800$$

-4800

80 $\quad -60$

$$x^2 + 20x - 4800 = 0$$

$$(x+80)(x-60) = 0$$

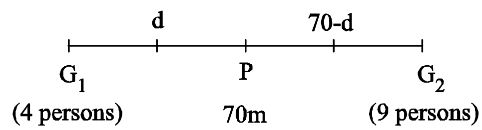
$x + 80 = 0$	$x - 60 = 0$
$x = -80$	$x = 60$
not possible	

$\therefore$ Speed of the passenger train = 60 km/hr

Speed of the express train $= 60 + 20$

$= 80$ km/hr

7. Music is been played is two opposite galleries with certain group of school. In the first gallery a group of 4 singers were singing and in the second gallery. 9 singers were singing. The two galleries are separated by the distance of 70m. Where should a person for hearing the same intensity of the singers voice? [Hint: The ratio of the sound intensity is equal to the square of the ratio of their corresponding distances).



Here $G_1 G_2 = 70$ m

Let $G_1 P =$ 'd' m we get $G_2 P = (70 - d)$ m.

Since the ratio of the sound intensity is equal to the square of the ratio of their corresponding distances.

$$\therefore \frac{4}{9} = \frac{d^2}{(70-d)^2}$$

$$\left(\frac{2}{3}\right)^2 = \left(\frac{d}{70-d}\right)^2$$

$$\frac{2}{3} = \frac{d}{70-d}$$

$$2(70-d) = 3d$$

$$140 - 2d = 3d$$

$$140 = 3d + 2d$$

$$140 = 5d$$

$$\frac{140}{5} = d$$

$$\boxed{28 = d}$$

$\therefore G_1 P = 28$ m

$G_2 P = 70 - 28$

$= 42$ m

(or) (Another method)

$$4(70-d)^2 = 9d^2$$

$$4\,(4900 - 140d + d^2) = 9d^2$$
$$19600 - 560d + 4d^2 = 9d^2$$
$$5d^2 + 560d - 19600 = 0$$
$$d^2 + 112d - 3920 = 0$$
$$(d + 140)\,(d - 28) = 0$$

$d + 140 = 0$	$d - 28 = 0$
$d = -140$	$d = 28$
not possible	

$\therefore$ The person should stand 28m from gallery I (or) 42 m from gallery II.

Exercise 3.13

Key points

Nature of roots of a quadratic equations:

Values of discriminant $\Delta = b^2 - 4ac$	**Nature of roots**
$\Delta > 0$	Real and unequal roots
$\Delta = 0$	Real and equal roots
$\Delta < 0$	No real roots

Type I: Nature of roots based sums
Q.No.1(i)(ii)(iii)(iv)(v), Example 3.41(i)(ii)(iii)

1. Determine the nature of the roots for the following quadratic equations

(i) $15x^2 + 11x + 2 = 0$

$a = 15\ ;\ b = 11\ ;\ c = 2$

$$\Delta = b^2 - 4ac$$
$$= (11)^2 - 4\,(15)\,(2)$$
$$= 121 - 120$$
$$= 1 > 0$$

$\therefore$ Given roots are real and unequal

(ii) $x^2 - x - 1 = 0$

$a = 1,\ b = -1,\ c = -1$

$$\Delta = b^2 - 4ac$$
$$= (-1)^2 - 4\,(1)\,(-1)$$
$$= 1 + 4$$
$$= 5 > 0$$

Given roots are real and unequal.

(iii) $\sqrt{2}\,t^2 - 3t + 3\sqrt{2} = 0$

$a = \sqrt{2},\ b = -3,\ c = 3\sqrt{2}$

$$\Delta = b^2 - 4ac$$
$$= (-3)^2 - 4\,(\sqrt{2})\,(3\sqrt{2})$$
$$= 9 - 24$$
$$= -15 < 0$$

$\therefore$ No real roots

(iv) $9y^2 - 6\sqrt{2}\,y + 2 = 0$

$a = 9\ ;\ b = -6\sqrt{2}\ ;\ c = 2$

$$\Delta = b^2 - 4ac$$
$$= (-6\sqrt{2})^2 - 4\,(9)\,(2)$$
$$= 72 - 72$$
$$\Delta = 0$$

$\therefore$ roots are real and equal.

(v) $9a^2b^2x^2 - 24abcdx + 16c^2d^2 = 0,\ a \neq 0,\ b \neq 0$

$$A = 9a^2b^2$$
$$B = -24abcd$$
$$C = 16c^2d^2$$

$$\Delta = B^2 - 4AC$$
$$= (-24abcd)^2 - 4\,(9a^2b^2)\,(16c^2d^2)$$
$$= 576a^2b^2c^2d^2 - 576a^2b^2c^2d^2$$
$$\Delta = 0$$

$\therefore$ roots are real and equal.

Example 3.48

Determine the nature of roots for the following quadratic equations.

(i) $\boldsymbol{x^2 - x - 20 = 0}$

$a = 1\ ;\ b = -1\ ;\ c = -20$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-1)^2 - 4\,(1)\,(-20)\\ &= 1 + 80\\ &= 81 > 0\end{aligned}$$

$\therefore$ Roots are real and unequal.

(ii) $\boldsymbol{9x^2 - 24x + 16 = 0}$

$a = 9\ ;\ b = -24\ ;\ c = 16$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-24)^2 - 4\,(9)\,(16)\\ &= 576 - 576\\ \Delta &= 0\end{aligned}$$

$\therefore$ Roots are real and equal.

(iii) $\boldsymbol{2x^2 - 2x + 9 = 0}$

$a = 2\ ;\ b = -2\ ;\ c = 9$

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-2)^2 - 4\,(2)\,(9)\\ &= 4 - 72\\ \Delta &= -68 < 0\end{aligned}$$

$\therefore$ No real roots.

Type II: Find the value of '*k*'

Q.No. 2(i)(ii), Example 3.42(i)(ii)

2. Find the value(s) of '*k*' for which the roots of the following equations are real and equal. (i) $\boldsymbol{(5k - 6)\,x^2 + 2kx + 1 = 0}$

$a = 5k - 6$

$b = 2k$

$c = 1$

Given roots are real and equal

$$\begin{aligned}\Delta &= 0\\ b^2 - 4ac &= 0\\ (2k)^2 - 4\,(5k - 6)\,(1) &= 0\\ 4k^2 - 20k + 24 &= 0\\ k^2 - 5k + 6 &= 0\\ (k - 2)\,(k - 3) &= 0\end{aligned}$$

6

$-2 \quad -3$

$k - 2 = 0$	$k - 3 = 0$
$k = 2$	$k = 3$

(ii) $\boldsymbol{kx^2 + (6k + 2)\,x + 16 = 0}$

$a = k$

$b = 6k + 2$

$c = 16$

Given roots are real and equal

$$\begin{aligned}\Delta &= 0\\ b^2 - 4ac &= 0\\ (6k + 2)^2 - 4\,(k)\,(16) &= 0\\ 36k^2 + 24k + 4 - 64k &= 0\\ 36k^2 - 40k + 4 &= 0\\ 9k^2 - 10k + 1 &= 0\\ (k - 1)\,(9k - 1) &= 0\end{aligned}$$

9

$-9 \quad -1$

$\frac{-9}{9} \quad \frac{-1}{9}$

$-1 \quad \frac{-1}{9}$

$k - 1 = 0$	$9k - 1 = 0$
$\boldsymbol{k = 1}$	$\boldsymbol{k = \frac{1}{9}}$

Example 3.42

(i) Find the values of 'k' for which the quadratic equation $\boldsymbol{kx^2 - (8k + 4) + 81 = 0}$ ***has real and equal roots?***

$$kx^2 - (8k + 4) + 81 = 0$$

$a = k$

$b = -(8k + 4)$

$c = 81$

Given roots are real and equal

$\Delta = 0$

$$\boxed{b^2 - 4ac = 0}$$

$$[-(8k+4)]^2 - 4(k)(81) = 0$$

$$64k^2 + 64k + 16 - 324k = 0$$

$$64k^2 - 260k + 16 = 0$$

Divide by 4

$$16k^2 - 65k + 4 = 0$$

$$(k-4)(16k-1) = 0$$

64

$-64 \quad -1$

$\frac{-64}{16} \quad \frac{-1}{16}$

$-4 \quad \frac{-1}{16}$

$k - 4 = 0$ | $16k - 1 = 0$

$\mathbf{k = 4}$ | $\mathbf{k = \frac{1}{16}}$

(ii) Find the values of 'k' such that quadratic equation $(k+9)x^2 + (k+1)x + 1 = 0$ has no real roots?

$$(k+9)x^2 + (k+1)x + 1 = 0$$

$$a = k+9$$

$$b = k+1$$

$$c = 1$$

Given roots are no real roots

$$\therefore \quad \Delta < 0$$

$$b^2 - 4ac < 0$$

$$(k+1)^2 - 4(k+9)(1) < 0$$

$$k^2 + 2k + 1 - 4k - 36 < 0$$

$$k^2 - 2k - 35 < 0$$

$$(k-7)(k+5) < 0$$

$$\therefore \quad \mathbf{-5 < k < 7}$$

-35

$-7 \quad 5$

Note

If $\alpha < \beta$ and if $(x-\alpha)(x-\beta) < 0$ then $\alpha < x < \beta$.

Type III: (Prove the following)
Q.No. 3, 4, 5, Example 3.43

3. If the roots of $(a-b)x^2 + (b-c)x + (c-a) = 0$ are real and equal, then prove that b, a, c are in arithmetic progression.

$$(a-b)x^2 + (b-c)x + (c-a) = 0$$

$$A = a - b$$

$$B = b - c$$

$$C = c - a$$

Given roots are real and equal

$$\therefore \quad \Delta = 0$$

$$B^2 - 4AC = 0$$

$$(b-c)^2 - 4(a-b)(c-a) = 0$$

$$b^2 - 2bc + c^2 - 4(ac - a^2 - bc + ab) = 0$$

$$b^2 - 2bc + c^2 - 4ac + 4a^2 + 4bc - 4ab = 0$$

$$4a^2 + b^2 + c^2 - 4ab - 2bc - 4ac = 0$$

$$(-2a + b + c)^2 = 0$$

$$-2a + b + c = 0$$

$$2a = b + c$$

$$a = \frac{b+c}{2}$$

$\therefore \quad b, a, c$ are in A.P.

Note

If b, a, c are in A.P then

$$a = \frac{b+c}{2}$$

---

4. If a, b are real then show that the roots of the equation $(a-b)x^2 - 6(a+b)x - 9(a-b) = 0$ are real and unequal.

$$(a-b)x^2 - 6(a+b)x - 9(a-b) = 0$$

$$A = a - b$$

$$B = -6(a+b)$$

$$C = -9(a-b)$$

$\Delta = B^2 - B^2 - 4AC$

$= [-6(a+b)]^2 - 4(a-b)(-9)(a-b)$

$= 36(a+b)^2 + 36(a-b)(a-b)$

$= 36[(a+b)^2 + (a-b)^2]$

$= 36[2(a^2+b^2)]$

$= 72(a^2+b^2) > 0$

$a, b \in R$

Hence roots are real and unequal.

---

5. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + b^2 - ac = 0$ are real and equal prove that either $a = 0$ (or) $a^3 + b^3 + c^3 = 3abc$

$(c^2 - ab)x^2 - 2(a^2 - bc)x + b^2 - ac = 0$

$A = c^2 - ab$

$B = -2(a^2 - bc)$

$C = b^2 - ac$

Given roots are real and equal

$\therefore \quad \Delta = 0$

$B^2 - 4AC = 0$

$[-2(a^2 - bc)]^2 - 4(c^2 - ab)(b^2 - ac) = 0$

$4[a^2 - bc]^2 - 4[b^2c^2 - ac^3 - ab^3 + a^2bc] = 0$

divide by 4

$a^4 - 2a^2bc + b^2c^2 - b^2c^2 + ac^3 + ab^3 - a^2bc = 0$

$a^4 - 3a^2bc + ab^3 + ac^3 = 0$

$a[a^3 - 3abc + b^3 + c^3] = 0$

$\boxed{a = 0}$ or $a^3 - 3abc + b^3 + c^3 = 0$

$\boxed{a^3 + b^3 + c^3 = 3abc}$ proved

Example 3.43

Prove that the equation $x^2(p^2+q^2) + 2x(pr+qs) + r^2 + s^2 = 0$ has no real roots. If $ps = qr$, then show that the roots are real and equal.

$x^2(p^2+q^2) + 2x(pr+qs) + r^2 + s^2 = 0$

$A = p^2 + q^2$

$B = 2(pr + qs)$

$c = r^2 + s^2$

$\Delta = B^2 - 4AC$

$= [2(pr+qs)]^2 - 4(p^2+q^2)(r^2+s^2)$

$= 4[p^2r^2 + 2pqrs + q^2s^2]$
$\qquad - 4[p^2r^2 + p^2s^2 + q^2r^2 + q^2s^2]$

$= 4[p^2r^2 + 2pqrs + q^2s^2 - p^2r^2 - p^2s^2$
$\qquad - q^2r^2 - q^2s^2]$

$= 4[2pqrs - p^2s^2 - q^2r^2]$

$= -4[p^2s^2 - 2pqrs + q^2r^2]$

$= -4(ps - qr)^2 < 0 \qquad$ ...(1)

$\therefore$ since $\Delta = b^2 - 4ac < 0$, the roots are not real.

- If $ps = qr$ then

$\Delta = -4(ps - ps)^2$

$\Delta = 0$

Thus $\Delta = 0$ if $ps = qr$ and so the roots will be real and equal.

Exercise 3.14

Key points

1. The relation between roots and co-efficients of a quadratic equation.

Let α and β are the roots of the equation $ax^2 + bx + c = 0$, then

- $\alpha+\beta = \frac{-b}{a} = \frac{-\text{co. efficients of } x}{\text{co. efficient of } x^2}$
- $\alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{co}-\text{efficients of } x^2}$
- **Equation:**

$$x^2-(\alpha+\beta)x+\alpha\beta = 0$$

Note:

(i) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

(ii) $\alpha-\beta = \sqrt{(\alpha+\beta)^2-4\alpha\beta}$

(iii) $\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(iv) $\alpha^3-\beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

(v) $\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$

$= [(\alpha+\beta)^2-2\alpha\beta]^2 - 2(\alpha\beta)^2$

Type I: [Find the values of the following based on α, β]

Q.No. 1(i)(ii)(iii)(iv), 2(i)(ii)(iii), Example 3.45(i) to (vi), Example 3.46(i)(ii)

1. Write each of the following expression in terms of $\alpha+\beta$ and $\alpha\beta$

(i) $\frac{\alpha}{3\beta}+\frac{\beta}{3\alpha}$

$$= \frac{\alpha}{3\beta}+\frac{\beta}{3\alpha}$$

$$= \frac{3\alpha^2+3\beta^2}{9\alpha\beta}$$

$$= \frac{3(\alpha^2+\beta^2)}{9\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{3\alpha\beta}$$

(ii) $\frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha}$

$$= \frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha}$$

$$= \frac{\beta^2\alpha+\alpha^2\beta}{\alpha^3\beta^3}$$

$$= \frac{\alpha\beta(\beta+\alpha)}{(\alpha\beta)^3}$$

$$= \frac{\alpha+\beta}{(\alpha\beta)^2}$$

(iii) $(3\alpha-1)(3\beta-1)$

$$= (3\alpha-1)(3\beta-1)$$

$$= 9\alpha\beta-3\alpha-3\beta+1$$

$$= 9\alpha\beta-3(\alpha+\beta)+1$$

(iv) $\frac{\alpha+3}{\beta}+\frac{\beta+3}{\alpha}$

$$= \frac{\alpha+3}{\beta}+\frac{\beta+3}{\alpha}$$

$$= \frac{\alpha(\alpha+3)+\beta(\beta+3)}{\alpha\beta}$$

$$= \frac{\alpha^2+3\alpha+\beta^2+3\beta}{\alpha\beta}$$

$$= \frac{\alpha^2+\beta^2+3\alpha+3\beta}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2-2\alpha\beta+3(\alpha+\beta)}{\alpha\beta}$$

---

2. The roots of the equation $2x^2-7x+5=0$ are α and β without solving for the roots find

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$ **(ii)** $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ **(iii)** $\frac{\alpha+2}{\beta+2}+\frac{\beta+2}{\alpha+2}$

Given

$$2x^2-7x+5=0$$

$a=2$; $b=-7$; $c=5$

- Sum of roots $= \frac{-b}{a}$

$$\boxed{\alpha+\beta = \frac{7}{2}}$$

- Product of roots $= \frac{c}{a}$

$$\boxed{\alpha\beta = \frac{5}{2}}$$

(i) $\frac{1}{\alpha}+\frac{1}{\beta}$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha\beta}$$

$$=\frac{\frac{7}{2}}{\frac{5}{2}}$$

$$=\frac{7}{5}$$

(ii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{\left(\frac{7}{2}\right)^2-2\left(\frac{5}{2}\right)}{\frac{5}{2}}$$

$$=\left(\frac{49}{4}-5\right)\times\frac{2}{5}$$

$$=\frac{29}{4}\times\frac{2}{5}$$

$$=\frac{29}{10}$$

(iii) $\frac{\alpha+2}{\beta+2}+\frac{\beta+2}{\alpha+2}$

$$=\frac{\alpha+2}{\beta+2}+\frac{\beta+2}{\alpha+2}$$

$$=\frac{(\alpha+2)^2+(\beta+2)^2}{(\beta+2)(\alpha+2)}$$

$$=\frac{\alpha^2+4\alpha+4+\beta^2+4\beta+4}{\alpha\beta+2\alpha+2\beta+4}$$

$$=\frac{(\alpha^2+\beta^2)+4(\alpha+\beta)+8}{\alpha\beta+2(\alpha+\beta)+4}$$

$$=\frac{[(\alpha+\beta)^2-2\alpha\beta]+4(\alpha+\beta)+8}{\alpha\beta+2(\alpha+\beta)+4}$$

$$=\frac{\left[\left(\frac{7}{2}\right)^2-2\left(\frac{5}{2}\right)\right]+4\left(\frac{7}{2}\right)+8}{\frac{5}{2}+2\left(\frac{7}{2}\right)+4}$$

$$=\frac{\left(\frac{49}{4}-5\right)+14+8}{\frac{5}{2}+7+4}$$

$$=\frac{\frac{29}{4}+22}{\frac{5}{2}+11}$$

$$=\frac{\frac{117}{4}}{\frac{27}{2}}$$

$$=\frac{117}{4}\times\frac{2}{27}$$

$$=\frac{117}{54}$$

Example 3.45

If α and β are the roots of $x^2+7x+10=0$ find the values of (i) $\alpha-\beta$ (ii) $\alpha^2+\beta^2$ (iii) $\alpha^3-\beta^3$ (iv) $\alpha^4+\beta^4$ (v) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (vi) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

$x^2+7x+10=0$

$a=1$; $b=7$; $c=10$

- Sum of roots $=\frac{-b}{a}$

$$\boxed{\alpha+\beta=-7}$$

- Product of roots $=\frac{c}{a}$

$$\boxed{\alpha\beta=10}$$

(i) $\alpha - \beta$

$$\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$$
$$= \sqrt{(-7)^2 - 4(10)}$$
$$= \sqrt{49-40}$$
$$= \sqrt{9}$$
$$\alpha - \beta = 3$$

(ii) $\alpha^2 + \beta^2$

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$
$$= (-7)^2 - 2(10)$$
$$= 49 - 20$$
$$= 29$$

(iii) $\alpha^3 - \beta^3$

$$\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$$
$$= (3)^3 + 3(10)(3)$$

[use $\alpha - \beta = 3$ from(i)]

$$= 27 + 90$$
$$= 117$$

(iv) $\alpha^4 + \beta^4$

$$\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2(\alpha\beta)^2$$

[use $\alpha^2 + \beta^2 = 29$ From (ii)]

$$= (29)^2 - 2(10)^2$$
$$= 841 - 200$$
$$= 641$$

(v) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$$
$$= \frac{29}{10}$$

[use $\alpha^2 + \beta = 29$ from (ii)]

(vi) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

$$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta}$$
$$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$
$$= \frac{(-7)^3 - 3(10)(-7)}{10}$$
$$= \frac{-343+210}{10}$$
$$= \frac{-133}{10}$$

Example 3.46

If α, β are the roots of the equation $3x^2 + 7x - 2 = 0$, find the values of (i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

$$3x^2 + 7x - 2 = 0$$
$$a = 3,\ b = 7,\ c = -2$$

- Sum of roots $= \frac{-b}{a}$

$$\alpha + \beta = \frac{-7}{3}$$

- Product of roots $= \frac{c}{a}$

$$\alpha\beta = \frac{-2}{3}$$

(i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$$
$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$
$$= \frac{\left(\frac{-7}{3}\right)^2 - 2\left(\frac{-2}{3}\right)}{\frac{-2}{3}}$$
$$= \left(\frac{49}{9} + \frac{4}{3}\right) \times \left(\frac{-3}{2}\right)$$
$$= \left(\frac{49+12}{9}\right) \times \left(\frac{-3}{2}\right)$$
$$= \frac{-61}{6}$$

(ii) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$

$$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$=\frac{\left(\frac{-7}{3}\right)^3-3\left(\frac{-2}{3}\right)\left(\frac{-7}{3}\right)}{\frac{-7}{3}}$$

$$=\left(\frac{-343}{27}-\frac{14}{3}\right)\times\left(\frac{-3}{7}\right)$$

$$=\left(\frac{-343-126}{27}\right)\times\left(\frac{-3}{7}\right)$$

$$=-\frac{469}{27}\times\frac{-3}{7}$$

$$=\frac{67}{9}$$

Type II: Form the equation based on roots α and β

Q.No. 3(i)(ii)(iii), Example 3.47(i)(ii)(iii)

3. The roots of the equation $x^2+6x-4=0$ are α and β. Find the quadratic equation whose roots are

(i) α^2 and β^2 (ii) $\dfrac{2}{\alpha}$ and $\dfrac{2}{\beta}$

(iii) $\alpha^2\beta$ and $\beta^2\alpha$

$x^2+6x-4=0$

$a=1\,;\,b=6\,;\,c=-4$

- Sum of roots $=\dfrac{-b}{a}$

$$\boxed{\alpha+\beta=-6}$$

- Product of roots $=\dfrac{c}{a}$

$$\boxed{\alpha\beta=-4}$$

(i) α^2 and β^2

Sum	**Product**
$=\alpha^2+\beta^2$	$=\alpha^2\beta^2$
$=(\alpha+\beta)^2-2\alpha\beta$	$=(\alpha\beta)^2$
$=(-6)^2-2(-4)$	$=(-4)^2$
$=36+8$	$=16$
$=44$	

$\therefore$ **Equation**

$$x^2-(\text{sum})\,x+\text{product}=0$$

$$x^2-44x+16=0$$

(ii) $\dfrac{2}{\alpha}$ and $\dfrac{2}{\beta}$

Sum	**Product**
$\dfrac{2}{\alpha}+\dfrac{2}{\beta}$	$\dfrac{2}{\alpha}\times\dfrac{2}{\beta}$
$\dfrac{2\beta+2\alpha}{\alpha\beta}$	$=\dfrac{4}{\alpha\beta}$
$=\dfrac{2(\alpha+\beta)}{\alpha\beta}$	$=\dfrac{4}{-4}$
$=\dfrac{2(-6)}{-4}$	$=-1$
$=3$	

Equation

$$x^2-(\text{sum})\,x+\text{product}=0$$

$$x^2-3x-1=0$$

(iii) $\alpha^2\beta$ and $\beta^2\alpha$

Sum	**Product**
$\alpha^2\beta+\beta^2\alpha$	$\alpha^2\beta\times\beta^2\alpha$
$=\alpha\beta(\alpha+\beta)$	$=(\alpha\beta)^3$
$=-4(-6)$	$=(-4)^3$
$=24$	$=-64$

Equation

$$x^2-(\text{sum})\,x+\text{product}=0$$

$$x^2-24x-64=0$$

Example 3.47

If α, β are the roots of the equation $2x^2 - x - 1 = 0$, then form the equation whose roots are

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ *(ii)* $\alpha^2 \beta, \beta^2 \alpha$ *(iii)* $2\alpha + \beta, 2\beta + \alpha$

Given

$$2x^2 - x - 1 = 0$$

$a = 2\ ;\ b = -1\ ;\ c = -1$

- Sum of roots $= \frac{-b}{a}$

$$\boxed{\alpha + \beta = \frac{1}{2}}$$

- Product of roots $= \frac{c}{a}$

$$\boxed{\alpha \beta = \frac{-1}{2}}$$

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$

Sum	**Product**
$= \frac{1}{\alpha} + \frac{1}{\beta}$	$= \frac{1}{\alpha} \times \frac{1}{\beta}$
$= \frac{\beta + \alpha}{\alpha \beta}$	$= \frac{1}{\alpha \beta}$
$= \frac{\frac{1}{2}}{\frac{-1}{2}}$	$= \frac{1}{\frac{-1}{2}}$
$= -1$	$= -2$

$\therefore$ **Equation**

$$x^2 - (\text{sum})\, x + \text{product} = 0$$

$$x^2 + x - 2 = 0$$

(ii) $\alpha^2 \beta, \beta^2 \alpha$

Sum	**Product**
$= \alpha^2 \beta + \beta^2 \alpha$	$= \alpha^2 \beta \times \beta^2 \alpha$
$= \alpha \beta (\alpha + \beta)$	$= \alpha^3 \beta^3$
$= \frac{-1}{2}\left(\frac{1}{2}\right)$	$= (\alpha \beta)^3$
$= \frac{-1}{4}$	$= \left(\frac{-1}{2}\right)^3$
	$= \frac{-1}{8}$

Equation

$$x^2 - (\text{sum})\, x + \text{product} = 0$$

$$x^2 + \frac{1}{4} x - \frac{1}{8} = 0$$

Multiply by 8

$$8x^2 + 2x - 1 = 0$$

(iii) $2\alpha + \beta, 2\beta + \alpha$

Sum

$$2\alpha + \beta + 2\beta + \alpha$$
$$= 3\alpha + 3\beta$$
$$= 3(\alpha + \beta)$$
$$= 3\left(\frac{1}{2}\right)$$
$$= \frac{3}{2}$$

Product

$$= (2\alpha + \beta)(2\beta + \alpha)$$
$$= 4\alpha\beta + 2\alpha^2 + 2\beta^2 + \alpha\beta$$
$$= 5\alpha\beta + 2(\alpha^2 + \beta^2)$$
$$= 5\alpha\beta + 2[(\alpha + \beta)^2 - 2\alpha\beta]$$
$$= 5\left(\frac{-1}{2}\right) + 2\left[\left(\frac{1}{2}\right)^2 - 2\left(\frac{-1}{2}\right)\right]$$
$$= \frac{-5}{2} + 2\left[\frac{1}{4} + 1\right]$$

Step 2: To solve $x^2+x+1=0$, subtract $x^2+x+1=0$ from $y=x^2+4x+3$

that is $y=x^2+4x+3 \quad (-)$

$0=x^2+x+1$

$y=3x+2$

The equation represent a straight line. Draw the graph of $y=3x+2$ by forming the table of values as below.

x	-2	-1	0	1	2
$3x$	-6	-3	0	3	6
2	2	2	2	2	2
$y=3x+2$	-4	-1	2	5	8

Step 3: Observe that the graph of $y=3x+2$ does not intersect or touch the graph of the parabola $y=x^2+4x+3$

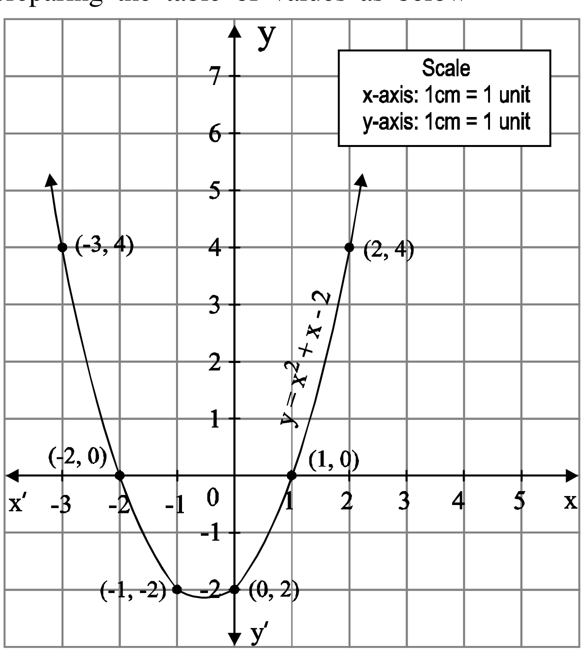
Thus $x^2+x+1=0$ has no real roots.

Example 3.51

Draw the graph of $y=x^2+x-2$ and hence solve $x^2+x-2=0$

Solution:

Step 1: Draw the graph of $y=x^2+x-2$ by preparing the table of values as below



x	-3	-2	-1	0	1	2
x^2	9	4	1	0	1	4
x	-3	-2	-1	0	1	2
-2	-2	-2	-2	-2	-2	-2
$y=x^2+x-2$	4	0	-2	-2	0	4

Step 2: To solve $x^2+x-2=0$, subtract $x^2+x-2=0$ from $y=x^2+x-2$

that is $y=x^2+x-2 \quad (-)$

$0=x^2+x+2$

$y=0$

The equation $y=0$ represents the X axis.

Step 3: Mark the point of intersection of the curve x^2+x-2 with the X axis. That is $(-2,0)$ and $(1,\ 0)$

Step 4: The x coordinates of the respective points form the solution set $\{-2,1\}$ for $x^2+x-2=0$

Example 3.52

Draw the graph of $y=x^2-4x+3$ and use it to solve $x^2-6x+9=0$

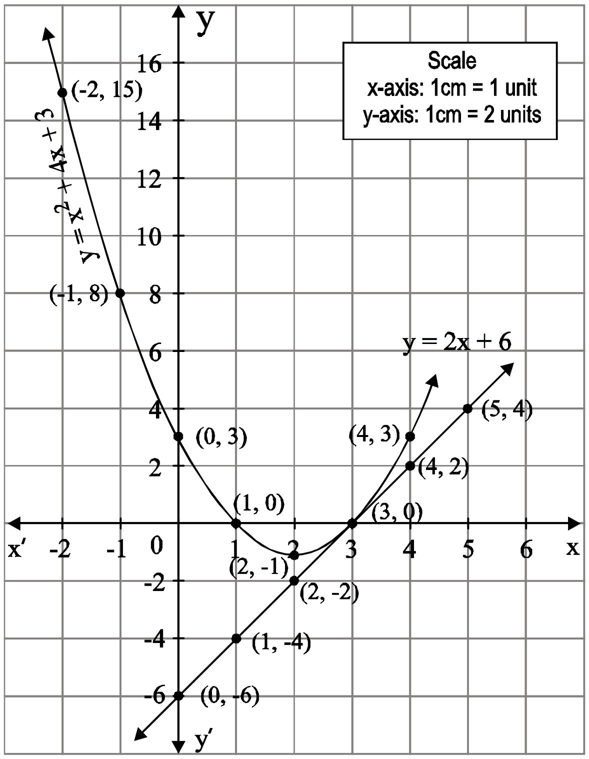
Solution:

Step 1: Draw the graph of $y=x^2-4x+3$ by preparing the table of values as below

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-4x$	8	4	0	-4	-8	-12	-16
3	3	3	3	3	3	3	3
$y=x^2-4x+3$	15	8	3	0	-1	0	3

Step 2:

To solve $x^2-6x+9=0$, subtract $x^2-6x+9=0$ from $y=x^2-4x+3$



that is $y = x^2 - 4x + 3$ $(-)$

$0 = x^2 - 6x + 9$

$y = 2x - 6$

The equation $y = 2x - 6$ represents a straight line. Draw the graph of $y = 2x - 6$ forming the table of values as below.

x	0	1	2	3	4	5
$2x$	0	2	4	6	8	10
-6	-6	-6	-6	-6	-6	-6
$y = 2x - 6$	-6	-4	-2	0	2	4

The line $y = 2x - 6$ intersect $y = x^2 - 4x + 3$ only at one point.

Step 3: Mark the point of intersection of the curve $y = x^2 - 4x + 3$ and $y = 2x - 6$ that is (3, 0)

Therefore, the x coordinate 3 is the only solution for the equation $x^2 - 6x + 9 = 0$

Exercise 3.15

1. Graph the following quadratic equations and state their nature of solutions

(i) $x^2 - 9x + 20 = 0$ (ii) $x^2 - 4x + 4 = 0$

(iii) $x^2 + x + 7 = 0$ (iv) $x^2 - 9 = 0$

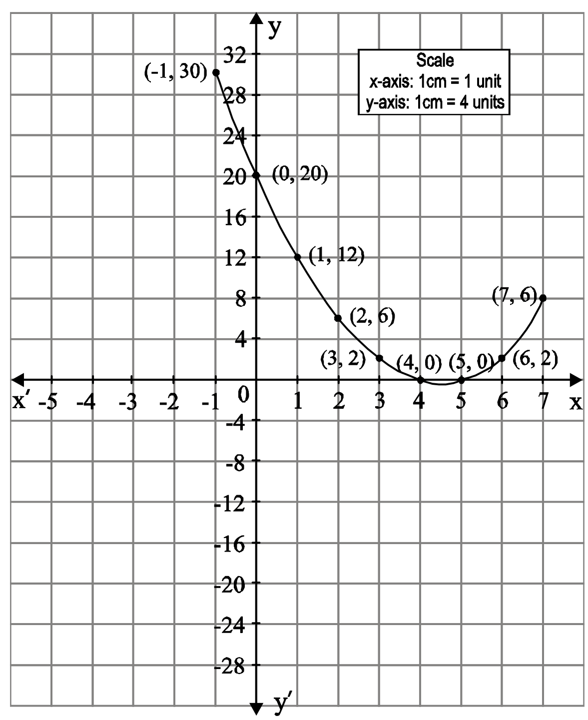
(v) $x^2 - 6x + 9 = 0$ (vi) $(2x - 3)(x + 2) = 0$

✍ **Solution:**

(i) $x^2 - 9x + 20 = 0$

Let $y = x^2 - 9x + 20$

X :	-2	-1	0	1	2	3	4	5	6	7
x^2 :	4	1	0	1	4	9	16	25	36	49
$-9x$:	18	9	0	-9	-18	-27	-36	-45	-54	-63
20 :	20	20	20	20	20	20	20	20	20	20
$y = x^2 - 9x + 20$:	42	30	20	12	6	2	0	0	2	6



- Plot the points $(-1, 30), (0, 20), (1, 12), (2, 6), (3, 2), (4, 0), (5, 0), (6, 2), (7, 6)$ on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y = x^2 - 9x + 20$.

- Here, the curve meets $x-$ axis at (4, 0), (5, 0).

  $\therefore$ The equation has real roots and $x-$coordinates of the points are $x=4, x=5$

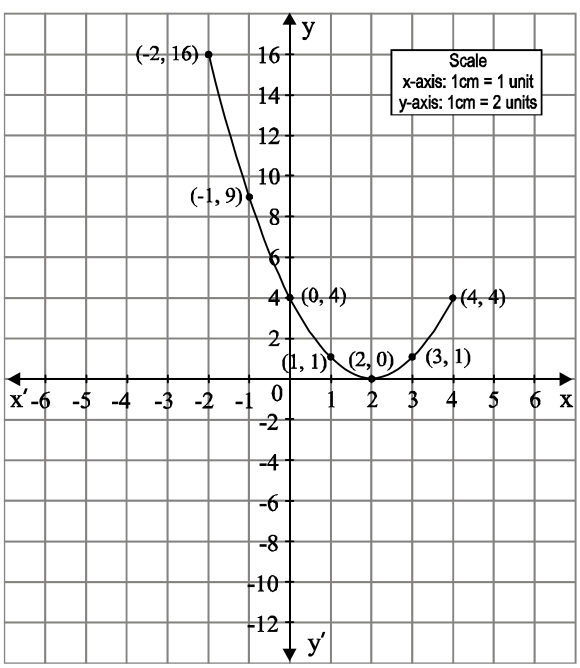
  $\therefore$ Solution $=\{4, 5\}$

(ii) $x^2-4x+4=0$

Let $y=x^2-4x+4$

x :	-2	-1	0	1	2	3	4
x^2 :	4	1	0	1	4	9	16
$-4x$:	8	4	0	-4	-8	-12	-16
4 :	4	4	4	4	4	4	4
$y=x^2-4x+4$:	16	9	4	1	0	1	4

- Plot the points $(-2, 16), (-1, 9), (0, 4), (1, 1), (2, 0), (3, 1), (4, 4)$ on the graph.



- Join all the points by a free-hand smooth curve. This curve is the graph of $y=x^2-4x+4$.
- Here, the curve meets $x-$ axis at (2, 0)

  $\therefore$ The equation 2 equal roots.

  $\therefore$ The $x-$coordinates of the points is $x=2$

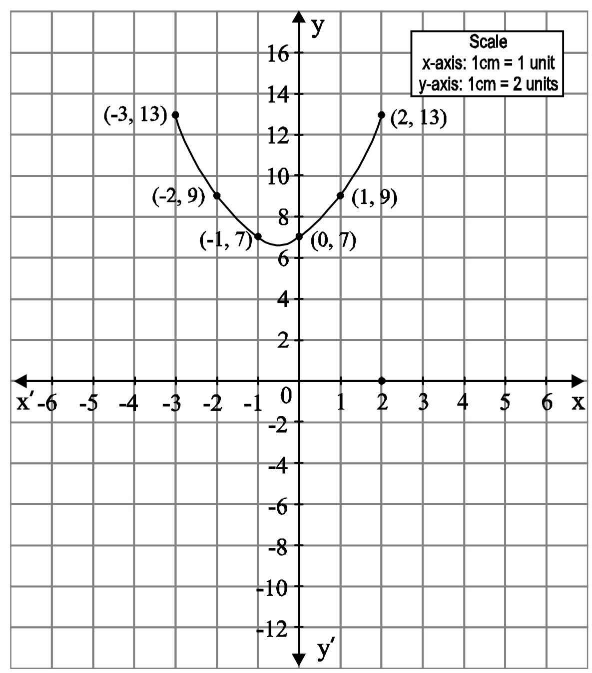
  $\therefore$ Solution $=\{2, 2\}$

(iii) $x^2+x+7=0$

Let $y=x^2+x+7$

x :	-3	-2	-1	0	1	2
x^2 :	9	4	1	0	1	4
x :	-3	-2	-1	0	1	2
7 :	7	7	7	7	7	7
$y=x^2+x+7$:	13	9	7	7	9	13

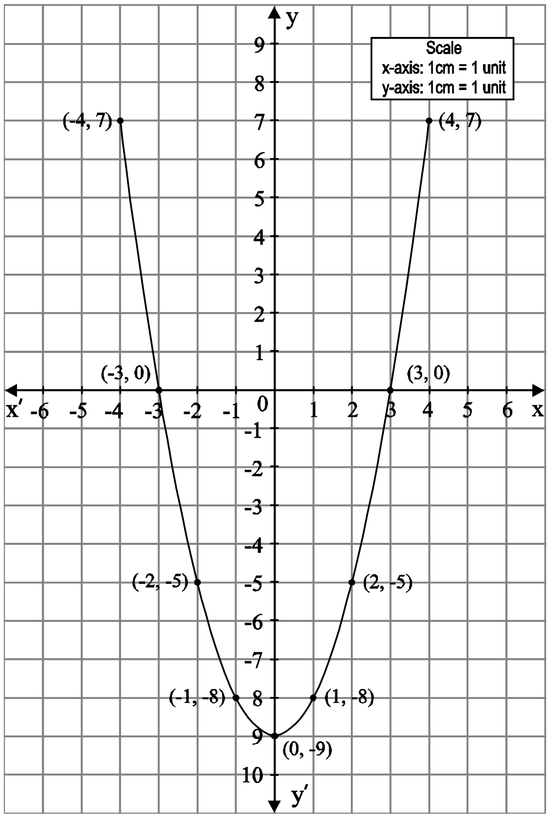
- Plot the points $(-3, 13), (-2, 9), (-1, 7), (0, 7), (1, 9), (2, 13), (3, 19)$ on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y=x^2+x+7$.
- Here, the curve does not meets the $x-$ axis and the curve has no real roots.



(iv) $x^2-9=0$

Let $y=x^2-9$

x :	-4	-3	-2	-1	0	1	2	3	4
x^2 :	16	9	4	1	0	1	4	9	16
-9 :	-9	-9	-9	-9	-9	-9	-9	-9	-9
$y=x^2-9$:	7	0	-5	-8	-9	-8	-5	0	7



- Plot the points $(-4, 7), (-3, 0), (-2, -5), (-1, -8), (0, -9), (1, -8), (2, -5), (3, 0), (4, 7)$ on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y = x^2 - 9$.
- Here, the curve meets $x-$ axis at 2 points $(-3, 0), (3, 0)$

  $\therefore$ The equation has real and unequal roots.

  $\therefore$ The $x-$ coordinates are $3, -3$ will be the solution.

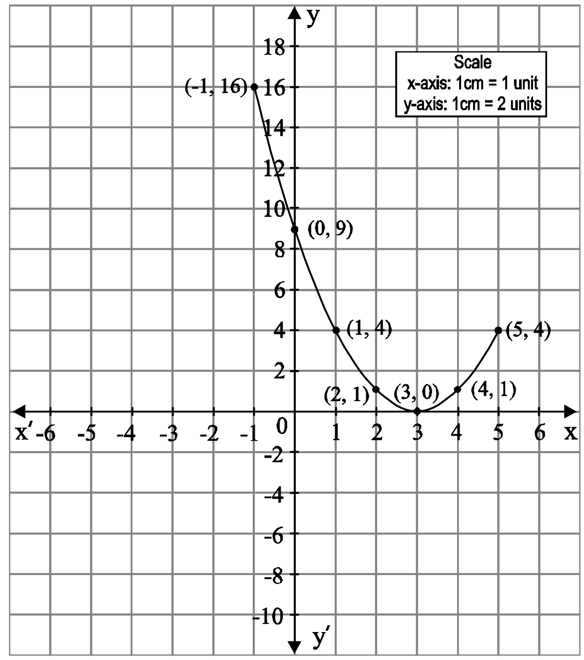
  $\therefore$ Solution $= \{-3, 3\}$

(v) $\boldsymbol{x^2 - 6x + 9 = 0}$

✍ **Solution:**

Let $y = x^2 - 6x + 9$

x :	-2	-1	0	1	2	3	4	5
x^2 :	4	1	0	1	4	9	16	25
$-6x$:	12	6	0	-6	-12	-18	-24	-30
9 :	9	9	9	9	9	9	9	9
$y = x^2 - 6x + 9$:	25	16	9	4	1	0	1	4



- Plot the points $(-1, 16), (0, 9), (1, 4), (2, 1), (3, 0), (4, 1), (5, 4)$ on the graph.
- Join all the points by a free-hand smooth curve. This curve is the graph of $y = x^2 - 6x + 9$.
- Here, the curve meets $x-$ axis at only one point (3, 0) and the equation has real and equal roots.

  $\therefore$ The $x-$ coordinate 3 will be the solution.

  $\therefore$ Solution $= \{3, 3\}$

(vi) $\boldsymbol{(2x - 3)(x + 2) = 0}$

$\boldsymbol{2x^2 + 4x - 3x - 6 = 0}$

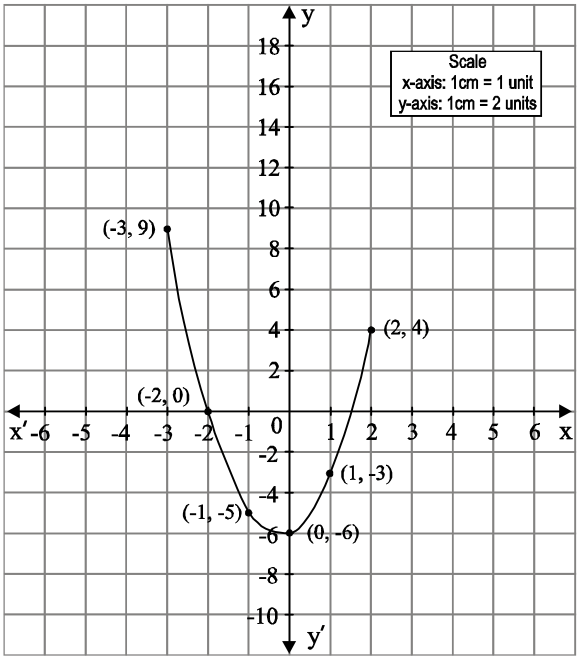
$\Rightarrow 2x^2 + x - 6 = 0$

✍ **Solution:**

Let $y = 2x^2 + x - 6$

x :	-3	-2	-1	0	1	2
$2x^2$:	18	8	2	0	2	8
x :	-3	-2	-1	0	1	2
-6 :	-6	-6	-6	-6	-6	-6
$y = 2x^2 + x - 6$:	9	0	-5	-6	-3	4

- Plot the points $(-3, 9), (-2, 0), (-1, -5), (0, -6), (1, -3), (2, 4)$ on the graph.



- Join all the points by a free-hand smooth curve. This curve is the graph of $y = 2x^2 + x - 6$.
- Here, the curve meets $x-$ axis at two points $(-2, 0)$, $(1.5, 0)$ and

$\therefore$ The equations has real and unequal roots.

$\therefore$ The $x-$ coordinates are $x = -2, -1.5$ will be the solution.

$\therefore$ Solution $= \{ -2, 3/2 \}$

---

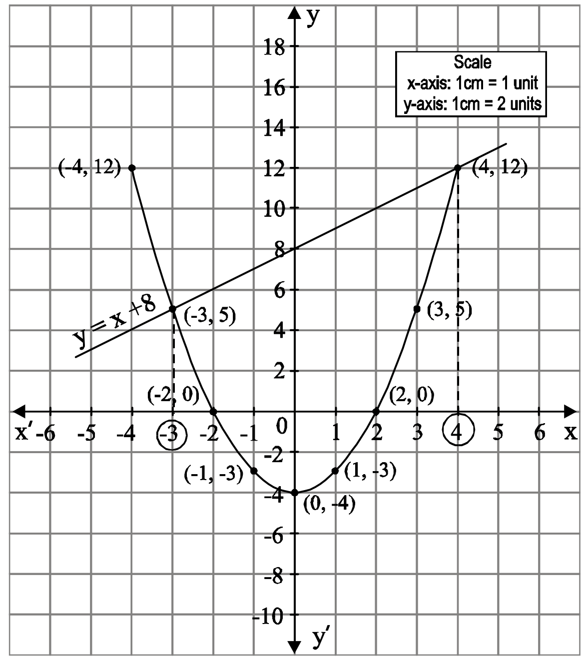
2. Draw the graph of $y = x^2 - 4$ and hence solve $x^2 - x - 12 = 0$

✍ **Solution:**

First, we draw the graph of $y = x^2 - 9$

x :	-4	-3	-2	-1	0	1	2	3	4
x^2 :	16	9	4	1	0	1	4	9	16
-4 :	-4	-4	-4	-4	-4	-4	-4	-4	-4
$y = x^2 - 4$:	12	5	0	-3	-4	-3	0	5	12

- Plot the points $(-4, 12), (-3, 5), (-2, 0), (-1, -3), (0, -4), (1, -3), (2, 0), (3, 5), (4, 12)$ on the graph.



- To solve $x^2 - x - 12 = 0$, subtract $x^2 - x - 12 = 0$ from $y = x^2 - 4$

from $y = x^2 - 4$

$$\begin{aligned} y &= x^2 + 0x - 4 \\ 0 &= x^2 - x - 12 \text{ sub} \\ \hline y &= x + 8 \end{aligned}$$

- We draw the graph of $y = x + 8$

x	-4	-3	-2	-1	0	1	2	3	4
y	4	5	6	7	8	9	10	11	12

- The line meets the curve at $(-3, 5), (4, 12)$

$\therefore$ The $x-$ coordinates $x = -3, x = 4$ will be the solution of $x^2 - x - 12 = 0$.

$\therefore$ Solution $= \{ -3, 4 \}$

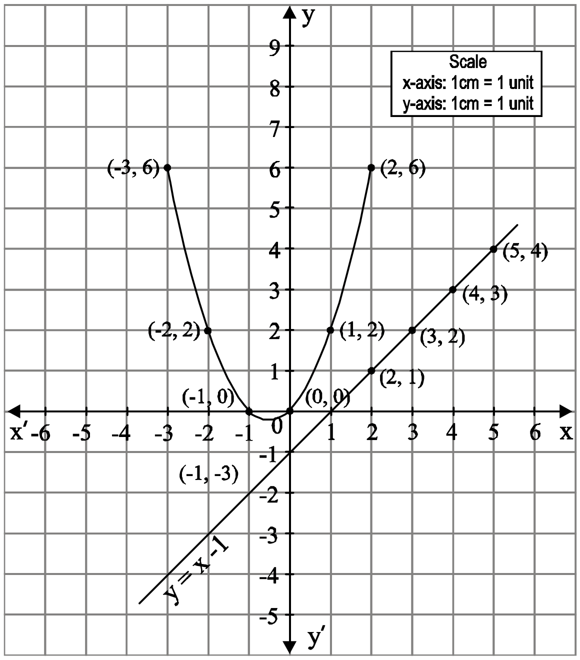
---

3. Draw the graph of $y = x^2 + x$ and hence solve $x^2 + 1 = 0$

✍ **Solution:**

First, we draw the graph of $y = x^2 + x$

x :	-3	-2	-1	0	1	2
x^2 :	9	4	1	0	1	4
x :	-3	-2	-1	0	1	2
$y = x^2 + x$:	6	2	0	0	2	6



- Plot the points $(-3, 6), (-2, 2), (-1, 0), (0, 0), (1, 2), (2, 6)$ on the graph.
- To solve $x^2 + 1 = 0$, subtract $x^2 + 1 = 0$ from $y = x^2 + x$

$$\begin{array}{r} y = x^2 + x \\ 0 = x^2 - 0x + 1 \text{ sub)} \\ \hline y = x - 1 \\ \hline \end{array}$$

- Draw the graph of $y = x - 1$

x	-4	-3	-2	-1	0	1	2	3	4	5
y	-5	-4	-3	-2	-1	0	1	2	3	4

- The line $y = x - 1$ does not meet the curve $y = x^2 + x$ and the equation has no real roots.

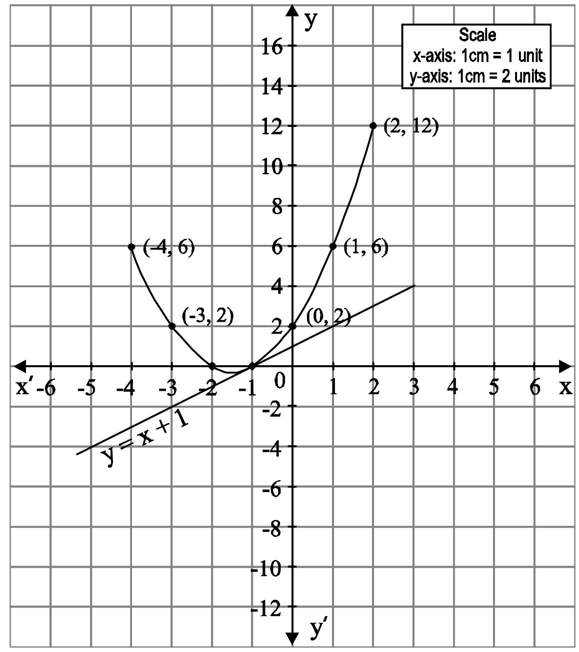
4. Draw the graph of $y = x^2 + 3x + 2$ and use it to solve $x^2 + 2x + 1 = 0$

✍ **Solution:**

First, we draw the graph of $y = x^2 + 3x + 2$

x :	-4	-3	-2	-1	0	1	2	3
x^2 :	16	9	4	1	0	1	4	9
$3x$:	-12	-9	-6	-3	0	3	6	9
2 :	2	2	2	2	2	2	2	2
$y = x^2 + 3x + 2$:	6	2	0	0	2	6	12	20

- Plot the points $(-4, 6), (-3, 2), (-2, 0), (-1, 0), (0, 2), (1, 6), (2, 12), (3, 20)$ on the graph.



- Join all the points to draw a free-hand smooth curve.
- To solve $x^2 + 2x + 1 = 0$, subtract $x^2 + 2x + 1 = 0$ from $y = x^2 + 3x + 2$

$$\begin{array}{r} y = x^2 + 3x + 2 \\ 0 = x^2 + 2x + 1 \text{ (sub)} \\ \hline y = x + 1 \\ \hline \end{array}$$

- Draw the graph of $y = x + 1$

x	-4	-3	-2	-1	0	1	2	3	4
y	-3	-2	-1	0	1	2	3	4	5

- The line $y = x + 1$ meets the curve $y = x^2 + 3x + 2$ at $(-1, 0)$ only and the equation $x^2 + 2x + 1 = 0$ has 2 equal roots.

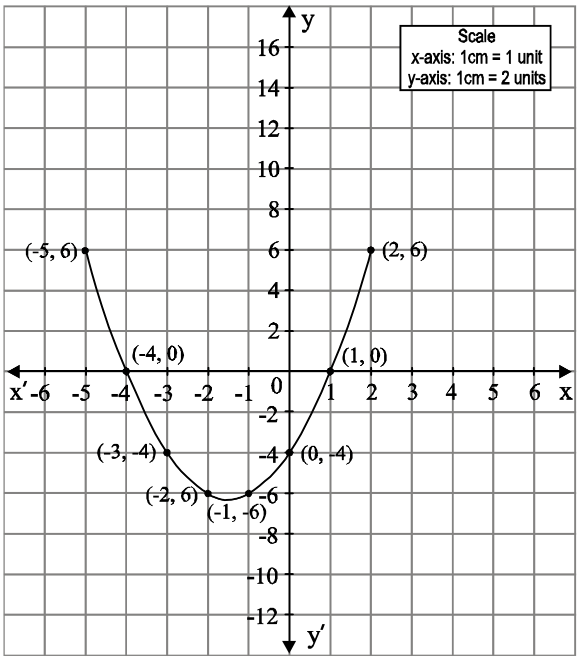
  $\therefore$ Solution $= \{-1, -1\}$

5. Draw the graph of $y = x^2 + 3x - 4$ and hence use it to solve $x^2 + 3x - 4 = 0$

✍ **Solution:**

First, we draw the graph of $y = x^2 + 3x - 4$

x :	-5	-4	-3	-2	-1	0	1	2	3
x^2 :	25	16	9	4	1	0	1	4	9
$3x$:	-15	-12	-9	-6	-3	0	3	6	9
-4 :	-4	-4	-4	-4	-4	-4	-4	-4	-4
$y = x^2 + 3x - 4$:	6	0	-4	-6	-6	-4	0	6	14



- Plot the points $(-4, 0), (-3, -4), (-2, -6), (-1, -6), (0, -4), (1, 0), (2, 6), (3, 14), (4, 24)$ on the graph.
- Join all the points to draw a free-hand smooth curve.
- To solve $x^2 + 3x - 4 = 0$, subtract $x^2 + 3x - 4 = 0$ from $y = x^2 + 3x - 4$

$$y = x^2 + 3x - 4$$
$$0 = x^2 + 3x - 4 \text{ (sub)}$$
$$y = 0$$

which is the equation of $x-$ axis.

- The curve meets $x-$ axis at $(-4, 0), (1, 0)$ and the x co-ordinates of the points $x = -4, x = 1$ will be the solution of $x^2 + 3x - 4 = 0$

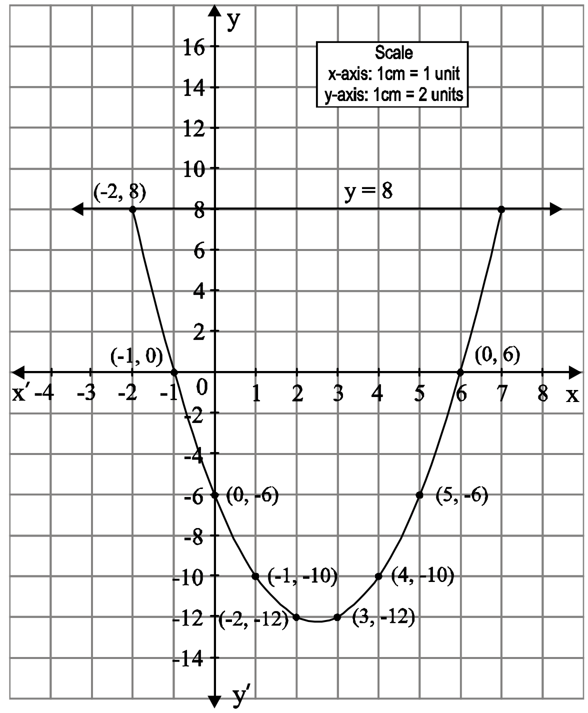
$\therefore$ Solution $= \{-4, 1\}$

6. Draw the graph of $y = x^2 - 5x - 6$ and hence solve $x^2 - 5x - 14 = 0$

✍ **Solution:**

First, we draw the graph of $y = x^2 - 5x + 6$

x :	-3	-2	-1	0	1	2	3	4	5	6	7
x^2 :	9	4	1	0	1	4	9	16	25	36	49
$-5x$:	15	10	5	0	-5	-10	-15	-20	-25	-30	-35
-6 :	-6	-6	-6	-6	-6	-6	-6	-6	-6	-6	-6
$y = x^2 - 5x + 6$:	18	8	0	-6	-10	-12	-12	-10	-6	0	8



- Plot the points and join them by a hand-free smooth curve.
- To solve $x^2 - 5x - 14 = 0$, subtract $x^2 - 5x - 14 = 0$ from $y = x^2 - 5x - 6$

$$y = x^2 - 5x - 6$$
$$0 = x^2 - 5x - 14 \text{ (sub)}$$
$$y = 8$$

a line parallel to $x-$ axis.

- The line $y = 8$ meets the curve $y = x^2 - 5x + 6$ at $(-2, 8), (7, 8)$

The x co-ordinates of the points $x = -2, x = 7$ will be the solution of $x^2 - 5x - 14 = 0$.

$\therefore$ Solution $= \{-2, 7\}$

7. Draw the graph of $y = 2x^2 - 3x - 5$ and hence solve $2x^2 - 4x - 6 = 0$

✍ **Solution:**

First, we draw the graph of $y = 2x^2 - 3x - 5$

x :	-2	-1	0	1	2	3
$2x^2$:	8	2	0	2	8	18
$-3x$:	6	3	0	-3	-6	-9
-5 :	-5	-5	-5	-5	-5	-5
$y = 2x^2 - 3x - 5$:	9	0	-5	-6	-3	4

Exercise 3.16

Key points

1. Addition and substraction of matrices

Two matrices can be added or subtracted if they have the same order. To add or subtract two matrices, simply add or subtract the corresponding elements.

2. Multiplication of matrix by a scalar

We can multiply the elements of the given matrix A by a non-zero number k to obtain a new matrix kA whose elements are multiplied by k. The matrix kA is called scalar multiplication of A.

Thus is $A = (a_{ij})_{m\times n}$ then, $kA = (ka_{ij})_{m\times n}$

for all $i = 1, 2, \ldots m$ and for

all $j = 1, 2, \ldots n$

Properties of Matrix Addition and Scalar Multiplication

Let A, B, C be $m \times n$ matrices and p and q be two non-zero scalars (numbers). Then we have the following properties.

(i) $A + B = B + A$	[Commutative property of matrix addition]
(ii) $A + (B + C) = (A + B) + C$	[Associative property of matrix addition]
(iii) $(pq)\,A = p\,(qA)$	[Associative property of scalar multiplication]
(iv) $IA = A$	[Scalar Identity property where I is the unit matrix]
(v) $p\,(A + B) = pA + pB$	[Distributive property of scalar and two matrices]
(vi) $(p + q)\,A = pA + qA$	[Distributive property of two scalars with a matrix]

Additive Identity

The null matrix or zero matrix is the identity for matrix addition.

Let A be any matrix.

Then, $A + O = O + A = A$ where O is the null matrix or zero matrix of same order as that of A.

Additive Inverse

If A be any given matrix then $-A$ is the additive inverse of A.

In fact we have $A + (-A) = (-A) + A = O$

Type I: Addition, subtraction based sums
Q.No. 1, 2, 4, Example 3.63

1. If $A = \begin{bmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{bmatrix}, B = \begin{bmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{bmatrix}$ then verify that

(i) $A + B = B + A$

(ii) $A + (-A) = (-A) + A = 0$

$$A = \begin{bmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{bmatrix}, B = \begin{bmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{bmatrix}$$

- $A + B = \begin{bmatrix} 1+5 & 9+7 \\ 3+3 & 4+3 \\ 8+1 & -3+0 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 6 & 7 \\ 9 & -3 \end{bmatrix}$
- $B + A = \begin{bmatrix} 5+1 & 7+9 \\ 3+3 & 3+4 \\ 1+8 & 0-3 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 6 & 7 \\ 9 & -3 \end{bmatrix}$

Hence $A + B = B + A$ is verified.

$$A = \begin{bmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{bmatrix} \text{ then } -A = \begin{bmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{bmatrix}$$

- $\{A + (-A) = \begin{bmatrix} 1-1 & 9-9 \\ 3-3 & 4-4 \\ 8-8 & -3+3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$
- $(-A) + A = \begin{bmatrix} -1+1 & -9+9 \\ -3+3 & -4+4 \\ -8+3 & 3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Here $A + (-A) = (-A) + A = 0$ is verified.

2. If $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{bmatrix}$ **and** $c = \begin{bmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix}$ **then verify that** $A + (B + C) = (A + B) + C$

LHS: $A + (B + C)$

$$B + C = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 8 \\ 2 & 7 & 5 \\ -5 & 5 & -2 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 10 & 6 & 8 \\ 2 & 7 & 5 \\ -5 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 9 & 9 \\ 4 & 10 & -3 \\ -4 & 5 & -6 \end{bmatrix} \quad ...(1)$$

RHS $(A + B) + C$

$$A + B = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 5 \\ 3 & 12 & -6 \\ -6 & 1 & -5 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 6 & 6 & 5 \\ 3 & 12 & -6 \\ -6 & 1 & -5 \end{bmatrix} + \begin{bmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 9 & 9 \\ 4 & 10 & -3 \\ -4 & 5 & -6 \end{bmatrix} \quad ...(2)$$

From (1) and (2) $A + (B + C) = (A + B) + C$ is verified.

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4. If $A = \begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}, B = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$ **find the value of (i)** $B - 5A$ **(ii)** $3A - 9B$

(i) $B - 5A = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix} - 5\begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}$

$$= \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix} - \begin{pmatrix} 0 & 20 & 45 \\ 40 & 15 & 35 \end{pmatrix} = \begin{pmatrix} 7 & -17 & -37 \\ -39 & -11 & -26 \end{pmatrix}$$

(ii) $3A - 9B = 3\begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix} - 9\begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 12 & 27 \\ 24 & 9 & 21 \end{pmatrix} - \begin{pmatrix} 63 & 27 & 72 \\ 9 & 36 & 81 \end{pmatrix} = \begin{pmatrix} -63 & -15 & -45 \\ 15 & -27 & -60 \end{pmatrix}$$

Example 3.63

If $A = \begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix}, B = \begin{pmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{pmatrix}, C = \begin{bmatrix} 5 & 3 & 0 \\ -1 & -7 & 2 \\ 1 & 4 & 3 \end{bmatrix}$ ***compute the following:***

(i) $3A + 2B - C$ ***(ii)*** $\frac{1}{2}A - \frac{3}{2}B$

✍ **Solution:**

(i) $3A + 2B - C = 3\begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix} + 2\begin{pmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 3 & 0 \\ -1 & -7 & 2 \\ 1 & 4 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 24 & 9 \\ 9 & 15 & 0 \\ 24 & 21 & 18 \end{pmatrix} + \begin{pmatrix} 16 & -12 & -8 \\ 4 & 22 & -6 \\ 0 & 2 & 10 \end{pmatrix} + \begin{pmatrix} -5 & -3 & 0 \\ 1 & 7 & -2 \\ -1 & -4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 9 & 1 \\ 14 & 44 & -8 \\ 23 & 19 & 25 \end{pmatrix}$$

(ii) $\frac{1}{2}A - \frac{3}{2}B = \frac{1}{2}(A - 3B)$

$$= \frac{1}{2}\left(\begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix} - 3\begin{pmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{pmatrix}\right)$$

$$=\frac{1}{2}\left(\begin{pmatrix}1 & 8 & 3\\ 3 & 5 & 0\\ 8 & 7 & 6\end{pmatrix}+\begin{pmatrix}-24 & 18 & 12\\ -6 & -33 & 9\\ 0 & -3 & -15\end{pmatrix}\right)$$

$$=\frac{1}{2}\begin{pmatrix}-23 & 26 & 15\\ -3 & -28 & 9\\ 8 & 4 & -9\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{23}{2} & 13 & \frac{15}{2}\\ -\frac{3}{2} & -14 & \frac{9}{2}\\ 4 & 2 & -\frac{9}{2}\end{pmatrix}$$

Type II: [Find the values of *x, y* based sums]

Q.No. 3, 6, 8, 5(i)(ii), 7, Example 3.62

3. Find X and Y if $X+Y=\begin{pmatrix}7 & 0\\ 3 & 5\end{pmatrix}$ and $X-Y=\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}$

$$X+Y=\begin{pmatrix}7 & 0\\ 3 & 5\end{pmatrix}$$

$$X-Y=\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix} \quad \text{(Add)}$$

$$2X=\begin{pmatrix}10 & 0\\ 3 & 9\end{pmatrix}$$

$$X=\begin{pmatrix}\frac{10}{2} & \frac{0}{2}\\ \frac{3}{2} & \frac{9}{2}\end{pmatrix}$$

$$\boxed{X=\begin{pmatrix}5 & 0\\ \frac{3}{2} & \frac{9}{2}\end{pmatrix}}$$

$$X+Y=\begin{pmatrix}7 & 0\\ 3 & 5\end{pmatrix}$$

(−) (+)

$$X-Y=\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix} \quad \text{(sub)}$$

$$2Y=\begin{pmatrix}4 & 0\\ 3 & 1\end{pmatrix}$$

$$Y=\begin{pmatrix}\frac{4}{2} & \frac{0}{2}\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix}$$

$$\boxed{Y=\begin{pmatrix}2 & 0\\ \frac{3}{2} & \frac{1}{2}\end{pmatrix}}$$

6. Find x and y if $x\begin{pmatrix}4\\ -3\end{pmatrix}+y\begin{pmatrix}-2\\ 3\end{pmatrix}=\begin{pmatrix}4\\ 6\end{pmatrix}$

$$x\begin{pmatrix}4\\ -3\end{pmatrix}+y\begin{pmatrix}-2\\ 3\end{pmatrix}=\begin{pmatrix}4\\ 6\end{pmatrix}$$

$$\begin{pmatrix}4x\\ -3x\end{pmatrix}+\begin{pmatrix}-2y\\ 3y\end{pmatrix}=\begin{pmatrix}4\\ 6\end{pmatrix}$$

$$\begin{pmatrix}4x-2y\\ -3x+3y\end{pmatrix}=\begin{pmatrix}4\\ 6\end{pmatrix}$$

$4x-2y=4$ $\qquad$ $-3x+3y=6$

divide by 2 $\qquad$ divide by 3

$2x-y=2$ ...(1) $\qquad$ $-x+y=2$ ...(2)

Solve (1) and (2)

$$2x-y=2$$
$$-x+y=2$$
$$x=4$$

Put $x=4$ in (2)

$$-4+y=2$$
$$y=2+4$$
$$\boxed{y=6}$$

$\therefore$ **Solution:**

$$x=4\,;\,y=6$$

8. Solve for x, y: $\begin{pmatrix}x^2\\ y^2\end{pmatrix}+2\begin{pmatrix}-2x\\ -y\end{pmatrix}=\begin{pmatrix}5\\ 8\end{pmatrix}$

$$\begin{pmatrix}x^2\\ y^2\end{pmatrix}+2\begin{pmatrix}-2x\\ -y\end{pmatrix}=\begin{pmatrix}5\\ 8\end{pmatrix}$$

$$\begin{pmatrix}x^2\\ y^2\end{pmatrix}+\begin{pmatrix}-4x\\ -2y\end{pmatrix}=\begin{pmatrix}5\\ 8\end{pmatrix}$$

$$\begin{pmatrix} x^2-4x \\ y^2-2y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$x^2-4x=5$	$y^2-2y=8$
$x^2-4x-5=0$	$y^2-2y-8=0$
$(x+1)(x-5)=0$	$(y-4)(y+2)=0$
$x+1=0$ $\quad x-5=0$	$y-4=0$ $\quad y+2=0$
$\mathbf{x=-1}$ $\quad \mathbf{x=5}$	$\mathbf{y=4}$ $\quad \mathbf{y=-2}$

5. Find the values of x, y, z if

(i) $\begin{pmatrix} x-3 & 3x-z \\ x+y+7 & x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix}$

Given matrices are equal. So comparing corresponding elements.

$x-3=1$	$3x-z=0$
$x=1+3$	$3(4)-z=0$
$\mathbf{x=4}$	$\mathbf{12=z}$

$$x+y+7=1$$
$$4+y+7=1$$
$$y=1-11$$
$$\boxed{y=-10}$$

(ii) $(x \quad y-z \quad z+3)+(y \quad 4 \quad 3)=(4 \quad 8 \quad 16)$

$(x \quad y-z \quad z+3)+(y \quad 4 \quad 3)=(4 \quad 8 \quad 16)$

$(x+y \quad y-z+4 \quad z+3+3)=(4 \quad 8 \quad 16)$

Comparing we get

$$x+y=4 \quad ...(1)$$
$$y-z+4=8 \quad ...(2)$$
$$z+6=16$$
$$z=16-6$$
$$\boxed{z=10}$$

$$(2) \Rightarrow y-10+4=8$$
$$y-6=8$$
$$y=8+6$$
$$\boxed{y=14}$$

$$(1) \Rightarrow x+14=4$$
$$x=4-14$$
$$\boxed{x=-10}$$

7. Find the non-zero values of x satisfying the matrix equation

$$x\begin{pmatrix} 2x & 2 \\ 3 & x \end{pmatrix} + 2\begin{pmatrix} 8 & 5x \\ 4 & 4x \end{pmatrix} = 2\begin{pmatrix} x^2+8 & 24 \\ 10 & 6x \end{pmatrix}$$

$$x\begin{pmatrix} 2x & 2 \\ 3 & x \end{pmatrix} + 2\begin{pmatrix} 8 & 5x \\ 4 & 4x \end{pmatrix} = 2\begin{pmatrix} x^2+8 & 24 \\ 10 & 6x \end{pmatrix}$$

$$\begin{pmatrix} 2x^2 & 2x \\ 3x & x^2 \end{pmatrix} + \begin{pmatrix} 16 & 10x \\ 8 & 8x \end{pmatrix} = \begin{pmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{pmatrix}$$

$$\begin{pmatrix} 2x^2+16 & 2x+10x \\ 3x+8 & x^2+8x \end{pmatrix} = \begin{pmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{pmatrix}$$

Comparing we get

$2x+10x=48$	(or) $3x+8=20$
$12x=48$	$3x=20-8$
$x=\frac{48}{2}$	$3x=2$
	$x=\frac{12}{3}$
$\mathbf{x=4}$	$\mathbf{x=4}$

Example 3.62

Find the value of a, b, c, d, x, y from the following matrix equation

$$\begin{pmatrix} d & 8 \\ 3b & a \end{pmatrix} + \begin{pmatrix} 3 & a \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 2a \\ b & 4c \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}$$

$$\begin{pmatrix} d & 8 \\ 3b & a \end{pmatrix} + \begin{pmatrix} 3 & a \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 2a \\ b & 4c \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}$$

$$\begin{pmatrix} d+3 & 8+a \\ 3b-2 & a-4 \end{pmatrix} = \begin{pmatrix} 2 & 2a+1 \\ b-5 & 4c \end{pmatrix}$$

Comparing corresponding elements we get

$d+3=2$	$8+a=2a+1$
$d=2-3$	$8-1=2a-a$
$\mathbf{d=-1}$	$\mathbf{7=a}$

$$3b-2=b-5$$

$$3b - b = -5 + 2$$

$$2b = -3$$

$$\boxed{b = \frac{-3}{2}}$$

$$a - 4 = 4c$$

$$7 - 4 = 4c$$

$$3 = 4c$$

$$\boxed{\frac{3}{4} = c}$$

$$\therefore \quad a = 7 \; ; b = \frac{-3}{2} \; ; c = \frac{3}{4} \; ; d = -1$$

Exercise 3.18

KEY POINTS

Multiplication of Matrices

To multiply two matrices, the number of columns in the first matrix must be equal to the number or row in the second matrix.

If the order of matrix A is $m \times n$ and B is $n \times p$ then the order of AB is $m \times p$

$$A_{m \times n} \qquad B_{n \times p}$$

$$AB_{m \times p}$$

Properties of multiplication of matrix

(a) Matrix multiplication is not commutative in general

$$AB \neq BA$$

(b) Matrix multiplication is distributive over matrix addition

(i) If A, B, C are $m \times n, n \times p$ and $n \times p$ matrices respectively then $A\,(B + C) = AB + AC$ (Right Distributive Property)

(ii) If A, B, C are $m \times n, m \times n$ and $n \times p$ matrices respectively then $(A + B)\,C = AC + BC$ (Left Distributive Property)

(c) Matrix multiplication is always associative

If A, B, C are $m \times n, n \times p$ and $p \times q$ matrices respectively then $(AB)\,C = A\,(BC)$

(d) Multiplication of a matrix by a unit matrix

If A is a square matrix of order $n \times n$ and I is the unit matrix of same order then $AI = IA = A$.

Note:

- If x and y are two real numbers such that $xy = 0$ then either $x = 0$ or $y = 0$. But the condition may not be true with respect to two matrices.
- $AB = 0$ does not necessarily imply that $A = 0$ or $B = 0$ or both $A, B = 0$
- If A and B are any two non-zero matrices then $(A + B)^2 \neq A^2 + 2AB + B^2$
- However if $AB = BA$ then $(A + B)^2 = A^2 + 2AB + B^2$

Type II: Multiplication of matrices with same order

Q.No. 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13
Example 3.65, 3.66, 3.69

1. Find the order of the product matrix AB if

	(i)	(ii)	(iii)	(iv)	(v)
Order of A	3×3	4×3	4×2	4×5	1×1
Order of B	3×3	3×2	2×2	5×1	1×3

(i) Order of $A = 3 \times 3$; Order of $B = 3 \times 3$

$\therefore$ Order of $AB = 3 \times 3$

(ii) Order of $A = 4 \times 3$; Order of $B = 3 \times 2$

Order of $AB = 4 \times 2$

(iii) Order of $A = 4 \times 2$; Order of $B = 2 \times 2$

Order of $AB = 4 \times 2$

(iv) Order of $A = 4 \times 5$; Order of $B = 5 \times 1$

Order of $AB = 4 \times 1$

(v) Order of $A = 1\times 1$; Order of $B = 1\times 3$

Order of $AB = 1\times 3$

2. If A is of order $p\times q$ and B is of order $q\times r$ what is the order of AB and BA?

$A_{p\times q}\,;\,B_{q\times r} = AB_{p\times r}$

$B_{q\times r}\,;\,A_{p\times q}$

Here BA is not possible.

3. A has 'a' rows and '$a+3$' columns B has 'b' rows and $17-b$ columns and if both products AB and BA exist, find a, b

$$A_{a\times(a+3)}\,;\,B_{b\times(17-b)}$$

Product AB is possible when

$$a+3=b$$

$$a-b=-3 \quad \text{...(1)}$$

Product BA is possible when

$$17-b=a$$

$$a+b=17 \quad \text{...(2)}$$

Solve (1) and (2)

$$a-b=-3$$
$$a+b=17$$
$$2a=14$$
$$a=\frac{14}{7}$$
$$\boxed{a=2}$$

$(2)\Rightarrow a+b=17$

$$2+b=17$$
$$b=17-2$$
$$\boxed{b=15}$$

4. If $A=\begin{pmatrix}2 & 5\\ 4 & 3\end{pmatrix}, B=\begin{pmatrix}1 & -3\\ 2 & 5\end{pmatrix}$ find AB and BA. Check if $AB=BA$?

- $AB=\begin{pmatrix}2 & 5\\ 4 & 3\end{pmatrix}\begin{pmatrix}1 & -3\\ 2 & 5\end{pmatrix}$

$$=\begin{pmatrix}2+10 & -6+25\\ 4+6 & -12+15\end{pmatrix}$$

$$=\begin{pmatrix}12 & 19\\ 10 & 3\end{pmatrix}$$

- $BA=\begin{pmatrix}1 & -3\\ 2 & 5\end{pmatrix}\begin{pmatrix}2 & 5\\ 4 & 3\end{pmatrix}$

$$=\begin{pmatrix}2-12 & 5-9\\ 4+20 & 10+15\end{pmatrix}$$

$$=\begin{pmatrix}-10 & -4\\ 24 & 25\end{pmatrix}$$

Here $AB\neq BA$

6. Show that the matrices $A=\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix} B=\begin{pmatrix}1 & -2\\ -3 & 1\end{pmatrix}$ satisfy commutative property $AB=BA$.

- $AB=\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & -2\\ -3 & 1\end{pmatrix}$

$$=\begin{pmatrix}1-6 & -2+2\\ 3-3 & -6+1\end{pmatrix}$$

$$=\begin{pmatrix}-5 & 0\\ 0 & -5\end{pmatrix}$$

- $BA=\begin{pmatrix}1 & -2\\ -3 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}$

$$=\begin{pmatrix}1-6 & 2-2\\ -3+3 & -6+1\end{pmatrix}$$

$$=\begin{pmatrix}-5 & 0\\ 0 & -5\end{pmatrix}$$

Hence $AB=BA$ verified.

Example 3.65

If $A=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}, B=\begin{pmatrix}2 & 0\\ 1 & 3\end{pmatrix}$ find AB and BA. Check if $AB=BA$

- $AB=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\begin{pmatrix}2 & 0\\ 1 & 3\end{pmatrix}$

$$=\begin{pmatrix}4+1 & 0+3\\ 2+3 & 0+9\end{pmatrix}$$

$$=\begin{pmatrix}5 & 3\\ 5 & 9\end{pmatrix}$$

- $BA = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 4+0 & 2+0 \\ 2+3 & 1+9 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 2 \\ 5 & 10 \end{pmatrix}$$

Here $AB \neq BA$

Example 3.66

If $A = \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2\sqrt{2} \\ -2 & 2 \end{bmatrix}$ Show that A and B satisfy commutative property with respect to matrix multiplication.

- $AB = \begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$

$$= \begin{pmatrix} 4+4 & 4\sqrt{2}-4\sqrt{2} \\ 2\sqrt{2}-2\sqrt{2} & 4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

- $BA = \begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$

$$= \begin{pmatrix} 4+4 & -4\sqrt{2}+4\sqrt{2} \\ -2\sqrt{2}+2\sqrt{2} & 4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

Here $AB = BA$ verified.

---

7. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ show that (i) $A(BC) = (AB)C$

(ii) $(A-B)C = AC - BC$

(iii) $(A-B)^T = A^T - B^T$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

(i) $A(BC) = (AB)C$

- $BC = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 8+0 & 0+0 \\ 2+5 & 0+10 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 7 & 10 \end{pmatrix}$$

- $A(BC) = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 7 & 10 \end{pmatrix}$

$$= \begin{pmatrix} 8+14 & 0+20 \\ 8+21 & 0+30 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 20 \\ 29 & 30 \end{pmatrix} \quad ...(1)$$

- $AB = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 4+2 & 0+10 \\ 4+3 & 0+15 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 10 \\ 7 & 5 \end{pmatrix}$$

- $(AB)C = \begin{pmatrix} 6 & 10 \\ 7 & 15 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 12+10 & 0+20 \\ 14+15 & 0+30 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 20 \\ 29 & 30 \end{pmatrix} \quad ...(2)$$

From (1) and (2) $A(BC) = (AB)C$ is verified.

(ii) $(A-B)C = AC - BC$

- $A - B = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}$

$$= \begin{pmatrix} -3 & 2 \\ 0 & -2 \end{pmatrix}$$

- $(A-B)C = \begin{pmatrix} -3 & 2 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} -6+2 & 0+4 \\ 0-2 & 0-4 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 4 \\ -2 & -4 \end{pmatrix} \quad ...(1)$$

- $AC = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 2+2 & 0+4 \\ 2+3 & 0+6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 \\ 5 & 6 \end{pmatrix}$$

- $BC = \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 8+0 & 0+0 \\ 2+5 & 0+10 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 7 & 10 \end{pmatrix}$$

- $AC - BC = \begin{pmatrix} 4 & 4 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 7 & 10 \end{pmatrix}$

$$= \begin{pmatrix} -4 & 4 \\ -2 & -4 \end{pmatrix} \quad ...(2)$$

From (1) & (2) $(A-B)\,C = AC - BC$ is verified.

(iii) $(A-B)^T = A^T - B^T$

- $A - B = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 1 & 5 \end{pmatrix}$

$$= \begin{pmatrix} -3 & 2 \\ 0 & -2 \end{pmatrix}$$

- $(A-B)^T = \begin{pmatrix} -3 & 0 \\ 2 & -2 \end{pmatrix} \quad ...(1)$

- $A^T = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}; B^T = \begin{pmatrix} 4 & 1 \\ 0 & 5 \end{pmatrix}$

$$A^T - B^T = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 1 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 \\ 2 & -2 \end{pmatrix} \quad ...(2)$$

From (1) & (2)

$(A-B)^T = A^T - B^T$ is verified.

---

8. If $A = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}, B = \begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{pmatrix}$ then show that $A^2 + B^2 = I$.

$$A^2 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + 0 & 0+0 \\ 0+0 & 0+\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \cos^2\theta \end{pmatrix}$$

$$B^2 = \begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{pmatrix}\begin{pmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\theta + 0 & 0+0 \\ 0+0 & 0+\sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sin^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$A^2 + B^2 = \begin{pmatrix} \cos^2\theta & 0 \\ 0 & \cos^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0+0 \\ 0+0 & \cos^2\theta + \sin^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A^2 + B^2 = I$ proved

---

9. If $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ prove that $A\,A^T = I$

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$A\,A^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A\,A^T = I$ verified

---

10. Verify that $A^2 = I$ when $A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix} \times \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 25-24 & -20+20 \\ 30-30 & -24+25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A^2 = I$ is verified

13. If $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ show that $A^2 - 5A + 7I_2 = 0$

- $A^2 = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{pmatrix}$

$= \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}$

- $5A = 5\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 15 & 5 \\ -5 & 10 \end{pmatrix}$

- $7I_2 = 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$

LHS

$\therefore A^2 - 5A + 7I_2$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence

$\therefore A^2 - 5A + 7I_2 = 0$

Proved

Example 3.69

If $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ -4 & 2 \end{pmatrix}$, $C = \begin{pmatrix} -7 & 6 \\ 3 & 2 \end{pmatrix}$ Verify that $A(B+C) = AB + AC$

LHS

$A(B+C)$

$B + C = \begin{pmatrix} 1 & 2 \\ -4 & 2 \end{pmatrix} + \begin{pmatrix} -7 & 6 \\ 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} -6 & 8 \\ -1 & 4 \end{pmatrix}$

$A(B+C) = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -6 & 8 \\ -1 & 4 \end{pmatrix}$

$= \begin{pmatrix} -6-1 & 8+4 \\ 6-3 & -8+12 \end{pmatrix}$

$= \begin{pmatrix} -7 & 12 \\ 3 & 4 \end{pmatrix}$ ...(1)

RHS

$AB + AC$

$AB = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -4 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1-4 & 2+2 \\ -1-12 & -2+6 \end{pmatrix}$

$= \begin{pmatrix} -3 & 4 \\ -13 & 4 \end{pmatrix}$

$AC = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -7 & 6 \\ 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} -7+3 & 6+2 \\ 7+9 & -6+6 \end{pmatrix}$

$= \begin{pmatrix} -4 & 8 \\ 16 & 0 \end{pmatrix}$

$\therefore AB + AC = \begin{pmatrix} -3 & 4 \\ -13 & 4 \end{pmatrix} + \begin{pmatrix} -4 & 8 \\ 16 & 0 \end{pmatrix}$

$= \begin{pmatrix} -7 & 12 \\ 3 & 4 \end{pmatrix}$ ...(2)

From (1) & (2) $A(B+C) = AB + AC$ is verified.

11. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Show that $A^2 - (a+d)A = (bc - ad)I_2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- $A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$

- $(a+d)A = a + d\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \begin{bmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{bmatrix}$$

$$= \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix}$$

LHS

$A^2 - (a+d)A$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+bd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix}$$

$$= \begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix}$$

$$= bc-ad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= (bc-ad)\, I_2$ **RHS**

Type II: (Multiplication with different order matrices)

Example 3.64, Example 3.70, 12, Example 3.68, 3.67

Example 3.64

If $A = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} & \mathbf{5} \end{pmatrix}, B = \begin{pmatrix} \mathbf{8} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} & \mathbf{1} \\ \mathbf{5} & \mathbf{3} & \mathbf{1} \end{pmatrix}$ ***find AB.***

Here Order of A is 2×3

Order of B is 3×3

Here AB is defined of order 2×3

$$AB = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 5 \end{pmatrix} \begin{pmatrix} 8 & 3 & 1 \\ 2 & 4 & 1 \\ 5 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8+4+0 & 3+8+0 & 1+2+0 \\ 24+2+25 & 9+4+15 & 3+1+5 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 11 & 3 \\ 51 & 28 & 9 \end{pmatrix}$$

Example 3.70

If $A = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$ ***and*** $B = \begin{pmatrix} \mathbf{2} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}$

Show that $(AB)^T = B^T A^T$

LHS

$(AB)^T$

- $AB = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 4 \\ 0 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 2-2+0 & -1+8+2 \\ 4+1+0 & -2-4+2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 9 \\ 5 & -4 \end{pmatrix}$$

- $(AB)^T = \begin{pmatrix} 0 & 5 \\ 9 & -4 \end{pmatrix}$ ...(1)

RHS

$B^T \cdot A^T$

$$B^T = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \end{pmatrix};\ A^T = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$B^T \cdot A^T = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-2+0 & 4+1+0 \\ -1+8+2 & -2-4+2 \end{pmatrix}$$

$$B^T \cdot A^T = \begin{pmatrix} 0 & 5 \\ 9 & -4 \end{pmatrix} \quad \text{...(2)}$$

From (1) & (2)

$(AB)^T = B^T \cdot A^T$ is verified.

12. If $A = \begin{pmatrix} \mathbf{5} & \mathbf{2} & \mathbf{9} \\ \mathbf{1} & \mathbf{2} & \mathbf{8} \end{pmatrix}; B = \begin{pmatrix} \mathbf{1} & \mathbf{7} \\ \mathbf{1} & \mathbf{2} \\ \mathbf{5} & \mathbf{-1} \end{pmatrix}$

Verify that $(AB)^T = B^T \cdot A^T$.

LHS

$(AB)^T$

$$AB = \begin{pmatrix} 5 & 2 & 9 \\ 1 & 2 & 8 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 1 & 2 \\ 5 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 5+2+45 & 35+4-9 \\ 1+2+40 & 7+4-8 \end{pmatrix}$$

$$AB = \begin{pmatrix} 52 & 30 \\ 43 & 3 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} 52 & 43 \\ 30 & 3 \end{pmatrix} \quad ...(1)$$

RHS

$B^T \cdot A^T$

$$B^T = \begin{pmatrix} 1 & 1 & 5 \\ 7 & 2 & -1 \end{pmatrix}, A^T = \begin{pmatrix} 5 & 1 \\ 2 & 2 \\ 9 & 8 \end{pmatrix}$$

$$B^T \cdot A^T = \begin{pmatrix} 1 & 1 & 5 \\ 7 & 2 & -1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 2 & 2 \\ 9 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 5+2+45 & 1+2+40 \\ 35+4-9 & 7+4-8 \end{pmatrix}$$

$$= \begin{pmatrix} 52 & 43 \\ 30 & 3 \end{pmatrix} \quad ...(2)$$

From (1) & (2) $(AB)^T = B^T \cdot A^T$ is verified.

Example 3.68

If $\mathbf{A = (1 -1\ 2), B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}}$ ***and*** $\mathbf{C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}}$ ***show that*** $\mathbf{(AB)\,C = A\,(BC)}$.

LHS

$(A\,B)\,C$

$$AB = (1 -1\ 2)_{1\times 3} \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}_{3\times 2}$$

$$= (1-2+2 \quad -1-1+6)$$

$$= (1 \quad 4)_{1\times 2}$$

$$(AB)\,C = (1\ 4)_{1\times 2} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}_{2\times 2}$$

$$= (1+8 \quad 2-4)$$

$$= (9 \quad -2)_{1\times 2} \quad ...(1)$$

RHS

$A\,(BC)$

$$BC = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}_{3\times 2} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}_{2\times 2}$$

$$= \begin{pmatrix} 1-2 & 2+1 \\ 2+2 & 4-1 \\ 1+6 & 2-3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{pmatrix}_{3\times 2}$$

$$A\,(BC) = (1 -1\ 2)_{1\times 3} \begin{pmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{pmatrix}_{3\times 2}$$

$$= (-1-4+14 \quad 3-3-2)$$

$$= (9 \quad -2)_{1\times 2} \quad ...(2)$$

From (1), (2) $(AB)\,C = A\,(BC)$ is verified.

Example 3.67

Solve $\mathbf{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

$$2x + y = 4 \quad ...(1)$$

$$x + 2y = 5 \quad ...(2)$$

$$(1)\times 2 \Rightarrow 4x + 2y = 8$$

$$(-)\ (-)\ (-)$$

$$(2) \Rightarrow x + 2y = 5$$

$$3x = 3$$

$$x = \frac{3}{3}$$

$$\boxed{x = 1}$$

Put $x = 1$ in (1)

$2\,(1) + y = 4$

$y = 4 - 2$

$$\boxed{y = 2}$$

$\therefore$ Solution $x = 1;\ y = 2$

$$BC = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0-0 & 1+0 \\ 0+2 & 2-5 \\ 0-4 & 0+10 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 2 & -3 \\ -4 & 10 \end{pmatrix}$$

$\therefore$ Ans (1) (i) and (ii) onl

y

UNIT EXERCISE 3

1. Solve

$$\frac{1}{3}(x+y-5) = y-z = 2x-11 = 9-(x+2z)$$

Solution:

Solve

$$\frac{1}{3}(x+y-5) = y-z = 2x-11 = 9-(x+2z)$$

Let

$$\frac{1}{3}(x+y-5) = y-z$$
$$x+y-5 = 3(y-z)$$
$$x+y-5 = 3y-3z$$
$$x+y-3y+3z = 5$$
$$x-2y+3z = 5 \quad \text{...(1)}$$

Let

$$y-z = 2x-11$$
$$y-z-2x = -11$$
$$-2x+y-z = -11 \quad \text{...(2)}$$

Let

$$2x-11 = 9-(x+2z)$$
$$2x-11 = 9-x-2z$$
$$2x+x+2z = 9+11$$
$$3x+2z = 20 \quad \text{...(3)}$$

Eliminate 'y' from (1) & (2)

$$(1) \Rightarrow x-2y+3z = 5$$
$$(2)\times 2 \Rightarrow -4x+2y-2z = -22$$
$$-3x+z = -17 \quad \text{..(4)}$$

Solve (3), (4)

$$3x+2z = 20$$
$$-3x+z = -617$$
$$3z = 3$$
$$\boxed{z = 1}$$

$$(3) \Rightarrow 3x+2(1) = 20$$
$$3x = 20-2$$
$$3x = 18$$
$$x = \frac{18}{3}$$
$$\boxed{x = 6}$$

$$(1) \Rightarrow x-2y+3z = 5$$
$$6-2y+3(1) = 5$$
$$9-2y = 5$$
$$9-5 = 2y$$
$$\frac{4}{2} = y$$
$$\boxed{2 = y}$$

$\therefore$ Ans. $x = 6; y = 2; z = 1$

2. One hundred and fifty students are admitted to a school. They are distributed over three sections A, B and C. If 6 students are shifted from section A to section C, the sections will have equal number of students. If 4 times of students of section C exceeds the number of students of section A by the number of students in section B, find the number of students in the three sections.

Solution:

Let no. of students in three sections A, B, C is x, y, z.

Given

$$x + y + z = 150 \quad ...(1)$$

$$x - 6 = z + 6$$

$$x - z = 6 + 6$$

$$x - z = 12 \quad ...(2)$$

$$4z = x + y \quad ...(3)$$

Put (3) in (1)

$$x + y + z = 150$$

$$4z + z = 150$$

$$5z = 150$$

$$z = \frac{150}{5}$$

$$\boxed{z = 30}$$

$$(2) \Rightarrow x - 30 = 12$$

$$x = 12 + 30$$

$$\boxed{x = 42}$$

$$(1) \Rightarrow 42 + y + 30 = 150$$

$$y = 150 - 72$$

$$\boxed{y = 78}$$

$\therefore$ Number of students in Sec - A = 42, Sec - B = 78, Sec - C = 30

---

3. In a three-digit number, when the tens and the hundreds digit are interchanged the new number is 54 more than three times the original number. If 198 is added to the number, the digits are reversed. The tens digit exceeds the hundreds digit by twice as that of the tens digit exceeds the unit digit. Find the original number.

✍ **Solution:**

Let unit place $= z$

tenth place $= y$

hundred place $= x$

$\therefore$ The number $= 100x + 10y + z$

The Reversed number $= 100z + 10y + x$

Given

$$100y + 10x + z = 3\,(100x + 10y + z) + 54$$

$$100y + 10x + z = 300x + 30y + 3z + 54$$

$$10x - 300x + 100y - 30y + z - 32 = 54$$

$$-290x + 70y - 2z = 54$$

divide by -2

$$145x - 36y + 2 = -27 \quad ...(1)$$

Given $100z + 10y + x = 100x + 10y + z + 198$

$$x - 100x + 100z - z = 198$$

$$-99x + 99z = 198$$

divide by 99

$$-x + z = 2 \quad ...(2)$$

Given $(y - x) = 2\,(y - z)$

$$y - x = 2y - 2z$$

$$-x + y - 2y + 2s = 0$$

$$-x - y + 2z = 0$$

$$x + y - 2z = 0 \quad ...(3)$$

Solve (1) & (3)

$$(1) \Rightarrow 145x - 35y + z = -27$$

$$(3) \times 35 \Rightarrow 35x + 35y - 70z = 0$$

$$180x - 69z = -27 \quad ...(4)$$

Solve (2) & (4)

$$(2) \times 69 \Rightarrow -69x + 69z = 138$$

$$(4) \Rightarrow 180x - 69z = -27$$

$$111x = 111$$

$$\boxed{x = 1}$$

$$2 \Rightarrow 1 + z = 2$$

$$z = 2 + 1$$

$$\boxed{z = 3}$$

$$(3) \Rightarrow x + y - 2z = 0$$

$$1 + y - 2\,(3) = 0$$

$$y - 5 = 0$$

$$\boxed{y = 5}$$

$\therefore$ The number is

$= 100x + 10y + z$

$= 100\,(1) + 10\,(5) + 3$

$= 100 + 50 + 3$

$= 153$

---

4. Find the least common multiple of $xy\,(k^2+1)+k\,(x^2+y^2)$ and $xy\,(k^2-1)+k\,(x^2-y^2)$

✍ **Solution:**

$xy\,(k^2+1)+k\,(x^2+y^2)$ and $xy\,(k^2-1)+k\,(x^2-y^2)$

- $xy\,(k^2+1)+k\,(x^2+y^2)$

$= k^2\,xy + xy + x^2\,k + y^2\,k$

$= k^2\,xy + y^2\,k + xy + x^2\,k$

$= yk\,(xk+y) + x\,(xk+y)$

$= (xk+y)\,(x+yk)$

- $xy\,(k^2-1)+k\,(x^2-y^2)$

$= k^2\,xy - xy + x^2\,k - y^2\,k$

$= k^2\,xy - y^2\,k - xy + x^2\,k$

$= ky\,(kx-y) + x\,(kx-y)$

$= (kx-y)\,(x+ky)$

$\therefore$ LCM $= (x+ky)\,(xk+y)\,(xk-y)$

$= (x+ky)\,(x^2\,k^2-y^2)$

---

5. Find the GCD of the following by division algorithm

$2x^4+13x^3+27x^2+23x+7,$

$x^3+3x^2+3x+1,\ x^2+2x+1$

GCD

$2x^4+13x^3+27x^2+23x+7,$

$x^3+3x^2+3x+1,\ x^2+2x+1$

$$
\begin{array}{r|l}
 & 2x+7 \\
\hline
x^3+3x^2+3x+1 & 2x^4+13x^3+27x^2+23x+7 \\
 & (-) \\
 & 2x^4+6x^3+6x^2+2x \\
\hline
 & \quad 7x^3+21x^2+21x+7 \\
 & \quad 7x^3+21x^2+21x+7 \ \text{(sub)} \\
\hline
 & \qquad 0
\end{array}
$$

$$
\begin{array}{r|l}
 & x+1 \\
\hline
x^2+2x+1 & x^3+3x^2+3x+1 \\
 & (-) \\
 & x^3+2x^2+x \\
\hline
 & \quad x^2+2x+1 \\
 & \quad x^2+2x+1 \ \text{(sub)} \\
\hline
 & \qquad 0
\end{array}
$$

$\therefore$ GCD $= (x^2+2x+1) = (x+1)^2$

---

6. Reduce the given Rational expressions to its lowest form

(i) $\dfrac{x^{3a}-8}{x^{2a}+2x^a+4}$

(ii) $\dfrac{10x^3-25x^2+4x-10}{-4-10x^2}$

✍ **Solution:**

Simplify

(i) $\dfrac{x^{3a}-8}{x^{2a}+2x^a+4}$

Let $x^a = y$ then, we get

$$\frac{y^3-8}{y^2+2y+4}$$

$$= \frac{(y-2)\,(y^2+2y+4)}{y^2+2y+4}$$

$= y-2$

$= x^a - 2$

put $y = x^2$

(ii) $\dfrac{\mathbf{10x^3 - 25x^2 + 4x - 10}}{\mathbf{-4 - 10x^2}}$

- $10x^3 - 25x^2 + 4x - 10$

$$= 5x^2 (2x-5) + 2(2x-5)$$
$$= (2x-5)(5x^2+2)$$

- $-4 - 10x^2 \;= -2(2+5x^2)$

$$\therefore = \frac{(2x-5)(5x^2+2)}{-2(2+5x^2)}$$
$$= \frac{5-2x}{2}$$

= **7. Simplify** $\dfrac{\dfrac{1}{p}+\dfrac{1}{q+r}}{\dfrac{1}{p}-\dfrac{1}{q+r}} \times \left(1+\dfrac{q^2+r^2-p^2}{2qr}\right)$

Simplify:

$$\frac{\dfrac{1}{p}+\dfrac{1}{q+r}}{\dfrac{1}{p}-\dfrac{1}{q+r}} \times \left[1+\frac{q^2+r^2-p^2}{2qr}\right]$$

$$= \frac{\dfrac{q+r+P}{p(q+r)}}{\dfrac{q+r-p}{p(q+r)}} \times \left[\frac{2qr+q^2+r^2-p^2}{2qr}\right]$$

$$= \frac{p+q+r}{q+r-p} \times \left[\frac{(q+r)^2-p^2}{2qr}\right]$$

$$= \frac{p+q+r}{q+r-p} \times \frac{(q+r+p)(q+r-p)}{2qr}$$

$$= \frac{(p+q+r)^2}{2qr}$$

---

8. Arul, Ravi and Ram working together can clean a store in 6 hours. Working alone, Ravi takes twice as long to clean the store as Arul does. Ram needs three times as long as Arul does. How long would it take each if they are working alone?

✍ **Solution:**

Let x, y, z be the working speed of Arul, Ravi and Ram respectively.

W - be the total work done.

$$x + y + z = \frac{W}{6} \quad ...(1)$$

Given

- Ravi takes twice as Arul does

$$\frac{W}{y} = 2\left(\frac{W}{x}\right)$$
$$\frac{1}{y} = \frac{2}{x}$$
$$\boxed{y = \frac{x}{2}}$$

- Ram takes thrice as Arul does

$$\frac{W}{Z} = 3\left(\frac{W}{x}\right)$$
$$\frac{1}{z} = \frac{3}{x}$$
$$\boxed{z = \frac{x}{3}}$$

$$(1) \Rightarrow x + \frac{x}{2} + \frac{x}{3} = \frac{W}{6}$$
$$\frac{6x+3x+2x}{6} = \frac{W}{6}$$
$$\frac{11x}{6} = \frac{W}{6}$$
$$11x = W$$
$$\boxed{x = \frac{W}{11}}$$

- $y = \dfrac{x}{2} = \dfrac{W}{11 \times 2}$

$$\boxed{y = \frac{W}{22}}$$

- $z = \dfrac{x}{3} = \dfrac{W}{11 \times 3}$

$$\boxed{z = \frac{w}{33}}$$

$\therefore$ Arul above does $= \dfrac{W}{x} = \dfrac{W}{W/11} = 11$ hrs

Ravi above does $= \dfrac{W}{W/22} = 22$ hrs

Ram above does $= \dfrac{W}{2} = \dfrac{W}{W/33} = 33$ hrs

9. Find the square root of $289x^4 - 612x^3 + 970x^2 - 684x + 361$.

Solution:

Square root of

$$289x^4 - 612x^3 + 970x^2 - 684x + 361$$

$$\begin{array}{r|l}
 & 17 \quad -18 \quad 19 \\
\hline
17 & 289 \quad -612 \quad 970 \quad -684 \quad 361 \\
 & (-) \\
 & 289 \\
\hline
34-18 & \quad -612 \quad 970 \\
 & \quad (+) \quad (-) \\
 & \quad -612 \quad 324 \\
\hline
34-36 \quad 19 & \qquad 646 \quad -684 \quad 361 \\
 & \qquad (-) \quad (+) \quad (-) \\
 & \qquad 646 \quad -684 \quad 361 \\
\hline
 & \qquad 0
\end{array}$$

$$\therefore \sqrt{289x^4 - 612x^3 + 970x^2 - 684x + 361}$$

$$= |\, 17x^2 - 18x + 19 \,|$$

10. Solve $\sqrt{y+1} + \sqrt{2y-5} = 3$

Solution:

Solve: $\sqrt{y+1} + \sqrt{2y-5} = 3$

$$\sqrt{y+1} + \sqrt{2y-5} = 3$$

$$\sqrt{2y-5} = 3 - \sqrt{y+1}$$

Squaring on both sides

$$(\sqrt{2y-5})^2 = (3 - \sqrt{y+1})^2$$

$$2y - 5 = 9 - 6\sqrt{y+1} + y + 1$$

$$2y - 5 - 9 - y - 1 = -6\sqrt{y+1}$$

$$y - 15 = -6\sqrt{y+1}$$

Squaring on both sides

$$(y-15)^2 = (-6\sqrt{y+1})^2$$

$$y^2 - 30y + 225 = 36(y+1)$$

$$y^2 - 30y + 225 - 36y - 36 = 0$$

$$y^2 - 66y + 189 = 0$$

189: -3, -63

$$(y-3)(y-63) = 0$$

$$y - 3 = 0 \qquad y - 63 = 0$$

$$\mathbf{y = 3} \qquad \mathbf{y = 63}$$

11. A boat takes 1.6 hours longer to go 36 kms up a river than down the river. If the speed of the water current is 4 km per hr, what is the speed of the boat in still water?

Let speed of the boat in still water = 'x' km/hr

Distance = 36 km

Time difference $= 1.6$ hrs

$$= \frac{8}{5} \text{ hrs}$$

Given data

$$\frac{36}{x-4} - \frac{36}{x+4} = \frac{8}{5}$$

(Time taken to upstream - time taken to down stream)

$$36\left(\frac{1}{x-4} - \frac{1}{x+4}\right) = \frac{8}{5}$$

$$36\left(\frac{x+4-x+4}{(x-4)(x+4)}\right) = \frac{8}{5}$$

$$36\left(\frac{8}{x^2-16}\right) = \frac{8}{5}$$

$$\frac{36}{x^2-16} = \frac{1}{5}$$

$$x^2 - 16 = 1.80$$

$$x^2 = 180 + 16$$

$$x^2 = 196$$

$$x = \pm 14$$

Speed of the boat = 14 km/hr

(negative not possible)

12. Is it possible to design a rectangular park of perimeter 320 m and area 4800 m^2? If so find its length and breadth.

Solution:

Perimeter of a Rectangular Park $= 320$ m

Area $= 4800$ m^2

Let the dimension of the park length = 'x' m; breadth = 'y' m.

- Perimeter $= 320$ m

$$2(x+y) = 320$$
$$x+y = 160$$
$$y = 160 - x \quad \text{...(1)}$$

- Area $= 4800$ m^2

$$xy = 4800$$
$$x(160-x) = 4800$$
$$160x - x^2 = 4800$$
$$x^2 - 160x + 4800 = 0$$
$$(x-120)(x-40) = 0$$
$$x = 120$$
$$(1) \Rightarrow y = 160 - 120$$
$$= 40$$
$$x = 40$$
$$y = 160 - 40$$
$$= 120$$

$\therefore$ Length $= 120$ m

breadth $= 40$ m

13. At t minutes past 2 pm, the time needed to 3 pm is 3 minutes less than $\frac{t^2}{4}$. Find t.

Solution:

Time needed by the minutes hand show $\frac{t^2}{4} - 3$

Given data

$$\frac{t^2}{4} - 3 = 60 - t$$

-252 → 18, -14

$$t^2 - 12 = 240 - 4t$$
$$t^2 - 12 - 240 + 4t = 0$$
$$t^2 + 4t - 252 = 0$$
$$(t+18)(t-14) = 0$$

$t + 18 = 0$ $\qquad$ $t - 14 = 0$

$t = -18$ $\qquad$ $t = 14$ min

not possible

14. The number of seats in a row is equal to the total number of rows in a hall. The total number of seats in the hall will increase by 375 if the number of rows is doubled and the number of seats in each row is reduced by 5. Find the number of rows in the hall at the beginning.

Solution:

Let the number of rows be x

Number of seats in each row $= x$

$\therefore$ Total number of seats in the hall $= x \cdot x$

$= x^2$

Given

$$2x(x-5) = x^2 + 375$$
$$2x^2 - 10x = x^2 + 375$$
$$2x^2 - 10x - x^2 - 375 = 0$$
$$x^2 - 10x - 375 = 0$$
$$(x-25)(x+15) = 0$$
$$x - 25 = 0$$
$$x = 25$$
$$x + 15 = 0$$
$$x = -15$$

not possible

$\therefore$ Number of rows in the hall at the beginning is 25.

---

15. If α and β are the roots of the polynomial $f(x) = x^2 - 2x + 3$, find the polynomial whose roots are (i) $\alpha + 2, \beta + 2$ (ii) $\frac{\alpha - 1}{\alpha + 1}, \frac{\beta - 1}{\beta + 1}$.

✍ **Solution:**

$$f(x) = x^2 - 2x + 3$$

$a = 1; b = -2; c = 3$

Sum of roots $= \frac{-b}{a}$

$$\boxed{\alpha + \beta = 2}$$

Product of ratios $= \frac{c}{a}$

$$\boxed{\alpha\beta = 3}$$

(i) Given roots $\alpha + 2, \beta + 2$

Sum

$$(\alpha + 2) + (\beta + 2) = \alpha + \beta + 4$$
$$= 2 + 4$$
$$= 6$$

Product

$$= (\alpha + 2)(\beta + 2)$$
$$= \alpha\beta + 2\alpha + 2\beta + 4$$
$$= \alpha\beta + 2(\alpha + \beta) + 4$$
$$= 3 + 2(2) + 4$$
$$= 3 + 4 + 4$$
$$= 11$$

$\therefore$ **equation**

$x^2 -$ (sum) $x +$ product $= 0$

$$x^2 - 6x + 11 = 0$$

(ii) Given roots

$$\frac{\alpha - 1}{\alpha + 1}, \frac{\beta - 1}{\beta + 1}$$

Sum

$$\frac{\alpha - 1}{\alpha + 1} + \frac{\beta - 1}{\beta + 1}$$
$$= \frac{(\alpha - 1)(\beta + 1) + (\beta - 1)(\alpha + 1)}{(\alpha + 1)(\beta + 1)}$$
$$= \frac{\alpha\beta + \alpha - \beta - 1 + \alpha\beta + \beta - \alpha - 1}{\alpha\beta + \alpha + \beta + 1}$$
$$= \frac{2\alpha\beta - 2}{\alpha\beta + \alpha + \beta + 1}$$
$$= \frac{2(3) - 2}{3 + 2 + 1}$$
$$= \frac{4}{6}$$
$$= \frac{2}{3}$$

Product

$$\frac{\alpha - 1}{\alpha + 1} \times \frac{\beta - 1}{\beta + 1} = \frac{\alpha\beta - \alpha - \beta + 1}{\alpha\beta + \alpha + \beta + 1}$$
$$= \frac{\alpha\beta - (\alpha + \beta) + 1}{\alpha\beta + (\alpha + \beta) + 1}$$
$$= \frac{3 - 2 + 1}{3 + 2 + 1}$$
$$= \frac{2}{6}$$
$$= \frac{1}{3}$$

equation

$x^2 -$ (sum) $x +$ product $= 0$

$$x^2 - \frac{2}{3}x + \frac{1}{3} = 0$$

$$3x^2 - 2x + 1 = 0$$

16. If -4 is a root of the equation $x^2 + px - 4 = 0$ and if the equation $x^2 + px + q = 0$ has equal roots, find the values of p and q.

✍ **Solution:**

Given:

-4 is a root of $x^2 + px - 4 = 0$

$$\therefore (-4)^2 + p(-4) - 4 = 0$$

$$16 - 4p - 4 = 0$$

$$12 - 4p = 0$$

$$12 = 4p$$

$$\frac{12}{4} = p$$

$$\boxed{3 = p}$$

Given:

$x^2 + px + q = 0$

$\Rightarrow x^2 + 3x + q = 0$ has equal roots

- Sum of roots $= \frac{-b}{a}$

$$\alpha + \alpha = -3$$

$$2\alpha = -3$$

$$\alpha = \frac{-3}{2} \quad \text{...(1)}$$

- Product of roots $= \frac{c}{a}$

$$(\alpha)(\alpha) = q$$

$$\alpha^2 = q$$

$$\left(\frac{-3}{2}\right)^2 = q \quad \text{From (1)}$$

$$\boxed{\frac{9}{4} = q}$$

17. Two farmers Senthil and Ravi cultivates three varieties of grains namely rice, wheat and ragi. If the sale (in Rs) of three varieties of grains by both the farmers in the month of April is given by the matrix.

April sale in Rs

$$\begin{array}{cc} & \begin{array}{ccc}\text{rice} & \text{wheat} & \text{ragi}\end{array} \\ A = & \begin{pmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Senthil} \\ \text{Ravi} \end{array}$$

and the May month sale (in Rs.) is exactly twice as that of the April month sale for each variety.

(i) What is the average sales of the months April and May.

(ii) If the sales continues to increase in the same way in the successive months, what will be sales in the month of August?

✍ **Solution:**

April sale in Rs

$$\begin{array}{cc} & \begin{array}{ccc}\text{rice} & \text{wheat} & \text{ragi}\end{array} \\ A = & \begin{pmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{Senthil} \\ \text{Ravi} \end{array}$$

Given sale of male month is twice of April month

$$\therefore B = 2A = \begin{pmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{pmatrix}$$

(i) Average sales of April and May

$$\frac{A+B}{2} = \begin{pmatrix} \frac{1500}{2} & \frac{3000}{2} & \frac{4500}{2} \\ \frac{7500}{2} & \frac{4500}{2} & \frac{1500}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 750 & 1500 & 2250 \\ 3750 & 2250 & 750 \end{pmatrix}$$

(ii) Sales in the month of August

$$= 2 \times 2 \times 2 \times 2 \times A$$

$$= 16\begin{pmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{pmatrix}$$

$$= \begin{pmatrix} 8000 & 16000 & 24000 \\ 40000 & 24000 & 8000 \end{pmatrix}$$

18. If $\cos\theta\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\times\sin\theta\begin{pmatrix}x & -\cos\theta\\ \cos\theta & x\end{pmatrix}=I_2$, **find** x.

✍ **Solution:**

$$\cos\theta\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}+\sin\theta\begin{pmatrix}x & -\cos\theta\\ \cos\theta & x\end{pmatrix}=I_2$$

$$\begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \cos^2\theta\end{pmatrix}+\begin{pmatrix}x\sin\theta & -\sin\theta\cos\theta\\ \sin\theta\cos\theta & x\sin\theta\end{pmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{pmatrix}\cos^2\theta+x\sin\theta & 0\\ 0 & \cos^2\theta+x\sin\theta\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\cos^2\theta+x\sin\theta=1$$

$$x\sin\theta=1-\cos^2\theta$$

$$x\sin\theta=\sin^2\theta$$

$$x=\frac{\sin^2\theta}{\sin\theta}$$

$$\boxed{x=\sin\theta}$$

19. Given $A=\begin{pmatrix}p & 0\\ 0 & 2\end{pmatrix}, B=\begin{pmatrix}0 & -q\\ 1 & 0\end{pmatrix}, C=\begin{pmatrix}2 & -2\\ 2 & 2\end{pmatrix}$ **and if** $BA=C^2$, **find** p **and** q.

✍ **Solution:**

$$A=\begin{pmatrix}P & 0\\ 0 & 2\end{pmatrix}, B=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, C=\begin{pmatrix}2 & -2\\ 2 & 2\end{pmatrix}$$

Given $BA=C^2$

$$\begin{pmatrix}0 & -q\\ 1 & 0\end{pmatrix}\begin{pmatrix}p & 0\\ 0 & 2\end{pmatrix}=\begin{pmatrix}2 & -2\\ 2 & 2\end{pmatrix}\begin{pmatrix}2 & -2\\ 2 & 2\end{pmatrix}$$

$$\begin{pmatrix}0-0 & 0-2q\\ p+0 & 0+0\end{pmatrix}=\begin{pmatrix}4-4 & -4-4\\ 4+4 & -4+4\end{pmatrix}$$

$$\begin{pmatrix}0 & -2q\\ p & 0\end{pmatrix}=\begin{pmatrix}0 & -8\\ 8 & 0\end{pmatrix}$$

$$\boxed{P=8}$$

$$-2q=-8$$

$$q=\frac{8}{2}$$

$$\boxed{q=4}$$

20. $A=\begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix}, B=\begin{pmatrix}6 & 3\\ 8 & 5\end{pmatrix}, C=\begin{pmatrix}3 & 6\\ 1 & 1\end{pmatrix}$ **find the matrix** D**, such that** $CD-AB=0$

✍ **Solution:**

$A=\begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix}, B=\begin{pmatrix}6 & 3\\ 8 & 5\end{pmatrix}$ $C=\begin{pmatrix}3 & 6\\ 1 & 1\end{pmatrix}$ and $CD-AB=0$

Let $D=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

- $AB=\begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix}\begin{pmatrix}6 & 3\\ 8 & 5\end{pmatrix}$

$$=\begin{pmatrix}18+0 & 9+0\\ 24+40 & 12+25\end{pmatrix}=\begin{pmatrix}18 & 9\\ 4 & 37\end{pmatrix}$$

Given $CD-AB=0$

$$CD=AB$$

$$\begin{pmatrix}3 & 6\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}18 & 9\\ 64 & 37\end{pmatrix}$$

$$\begin{pmatrix}3a+6c & 3b+6d\\ a+c & b+d\end{pmatrix}=\begin{pmatrix}18 & 9\\ 64 & 37\end{pmatrix}$$

Comparing we get

$3a+6c=18$

$a+2c=6$ ...(1)

$(-)\ (-)\ (-)$

$a+c=64$ ...(2)

$c=-58$

$(2)\Rightarrow a+c=64$

$a-58=64$

$a=64+58$

$$\boxed{a=122}$$

- $3b+bd=9$

$b+2d=3$ ...(3)

$(-)\ (-)\ (-)$

$b+d=37$ ...(4)

$d=-34$

$b+d=37$

$b-34=37$

$b=37+34$

$$\boxed{b=71}$$

$$\therefore\ D=\begin{pmatrix}122 & 71\\ -58 & -34\end{pmatrix}$$

CHAPTER 4

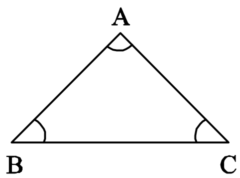
GEOMETRY

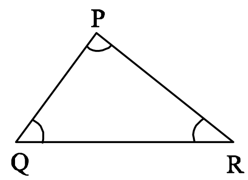
Exercise 4.1

KEY POINTS

I. Congruency and Similarity of Triangles

1. **Two triangles said to be congruent if they have same shape and same size.**





$\Delta ABC \equiv \Delta PQR$, then

* $\angle A = \angle P$ (Corresponding angles are equal).

$\angle B = \angle Q$

$\angle C = \angle R$

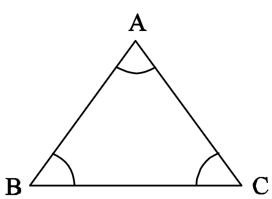
* $AB = PQ$ (Corresponding sides are equal)

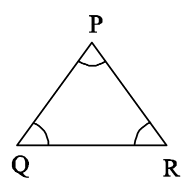
$BC = QR$

$CA = RP$

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = 1$$

2. Two triangles said to be **similar** if they same shape but not same size.





$\Delta ABC \sim \Delta PQR$, then

* $\angle A = \angle P$ (Corresponding angles are equal).

$\angle B = \angle Q$

$\angle C = \angle R$

* $AB \neq PQ$ (Corresponding sides are not equal).

$BC \neq QR$

$AC \neq PR$

- $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} > 1$ or < 1

Corresponding sides are proportional.

Criteria of Similarity

AA Criterion of similarity

In ΔABC and ΔPQR

if $\angle A = \angle P$ and $\angle B = \angle Q$ [any two corresponding angles are equal]

then $\Delta ABC \sim \Delta PQR$

SAS Criterian of similarity

In ΔABC and ΔPQR

if $\angle A = \angle P$ and

$\frac{AB}{PQ} = \frac{AC}{PR}$ then

(one corresponding angle two corresponding sides are equal)

$\Delta ABC \sim \Delta PQR$

SSS Criterian of Similarity

In ΔABC and ΔPQR

if $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

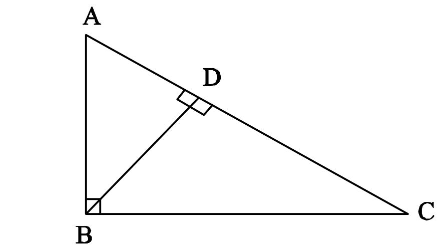
(Corresponding sides are proportional)

then $\Delta ABC \sim \Delta PQR$

Some results on similar triangles

1. ΔABC right angled at B, and $BD \perp AC$, then

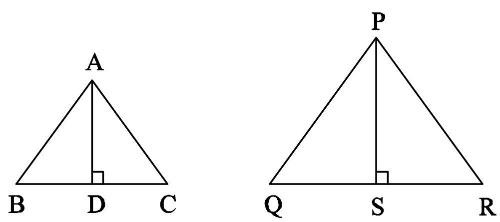
- $\Delta ADB \sim \Delta BDC$
- $\Delta ABC \sim \Delta ADB$
- $\Delta ABC \sim \Delta BDC$



2. If two triangles are similar, then the ratio of the corresponding sides are equal to the ratio of their corresponding altitudes and (Medians) also.

$$\Delta ABC \sim \Delta PQR, \text{ then}$$

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{AD}{PS}$$



3. If two triangles are similar, then the ratio of the corresponding sides are equal to the ratio of the corresponding perimeters.

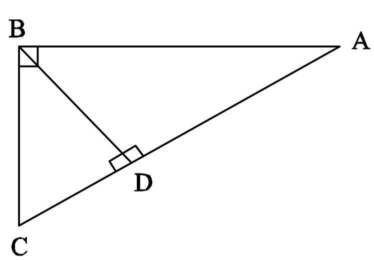
$$\Delta ABC \sim \Delta PQR \text{ then}$$

$$\frac{\text{Perimeter } \Delta ABC}{\text{Perimeter } \Delta PQR} = \frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$$

4. If two triangle are similar, then the area of two similar triangles are equal to the ratio of the squares of their corresponding sides.

$$\frac{\text{area } (\Delta ABC)}{\text{area } (\Delta PQR)} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$$

5. If two triangles have common vertex and their bases are on the same straight line, the ratio between their areas is equal to the ratio between the length of their bases.



$$\frac{\text{area } (\Delta ABD)}{\text{area } (\Delta BDC)} = \frac{AD}{DC}$$

6. Two triangles are said to be similar if their corresponding sides are proportional.

- Two triangles are equiangular if the corresponding angles are equal.
- If two triangles are similar, then they are equiangular.

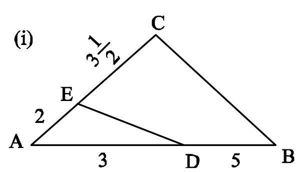
Exercise 4.1

Type I: Problems Based on Similarity

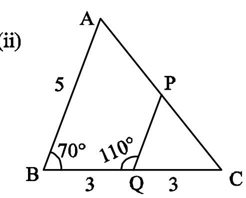
Q.No. 1(i)(ii), Example 4.1, 4.2, 4.3, 4.5, 4.6, 4.4, 2, 3, 5, 6, 8, 4.7, 4.8

1. Check whether the which triangles are similar and find the value of x.

(i)



(ii)



✍ **Solution:**

In ΔABC and ΔADE

(i) $\frac{AE}{AC} = \frac{2}{2+3.5} = \frac{2}{5.5} = \frac{4}{11}$

$\frac{AD}{AB} = \frac{3}{3+5} = \frac{3}{8}$

$\therefore \frac{AE}{AC} \neq \frac{AD}{AB}$

$\therefore$ the 2 triangles are not similar.

(ii) Given In ΔABC and ΔPQC

$\angle PQB = 110° \Rightarrow \angle PQC = 70° = \angle QBA$

$\therefore$ Corresponding angles are equal.

$$\therefore \quad \Delta ABC \sim \Delta PQC$$

$\Rightarrow \frac{AB}{PQ} = \frac{BC}{QC}$

$\Rightarrow \frac{5}{x} = \frac{6}{3}$

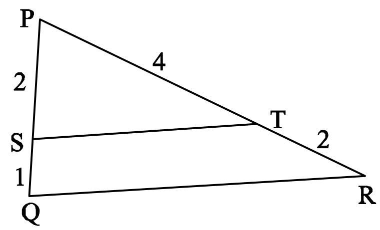
$\therefore \quad 2x = 5$

$x = 2.5$ cm

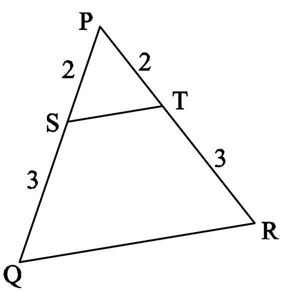
Example 4.1

Show that $\Delta PST \sim \Delta PQR$

(i)



(ii)



✍ **Solution:**

(i) In ΔPST and ΔPQR,

$$\frac{PS}{PQ} = \frac{2}{2+1} = \frac{2}{3}, \frac{PT}{PR} = \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3}$$

Thus, $\frac{PS}{PQ} = \frac{PT}{PR}$ and $\angle P$ is common

Therefore, by SAS similarity,

$$\Delta PST \sim \Delta PQR$$

(ii) In ΔPST and ΔPQR,

$$\frac{PS}{PQ} = \frac{2}{2+3} = \frac{2}{5}, \frac{PT}{PR} = \frac{2}{2+3} = \frac{2}{5}$$

Thus, $\frac{PS}{PQ} = \frac{PT}{PR}$ and $\angle P$ is common

Therefore, by SAS similarity,

$$\Delta PST \sim \Delta PQR$$

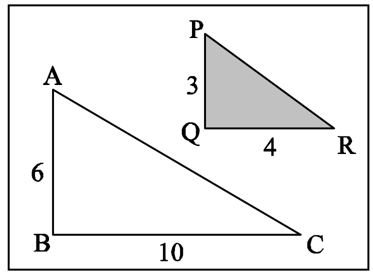
Example 4.2

Is $\Delta ABC \sim \Delta PQR$?

✍ **Solution:**

In ΔABC and ΔPQR,

$$\frac{PQ}{AB} = \frac{3}{6} = \frac{1}{2}, \frac{QR}{BC} = \frac{4}{10} = \frac{2}{5}$$



Since $\frac{1}{2} \neq \frac{2}{5}, \frac{PQ}{AB} \neq \frac{QR}{BC}$

The corresponding sides are not proportional.

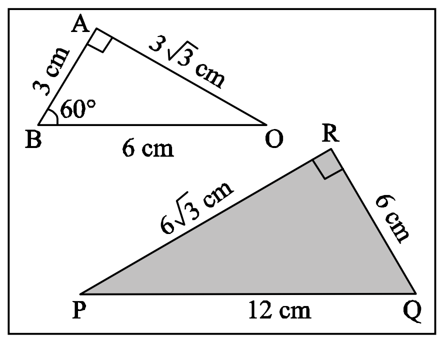
Therefore ΔABC is not similar to ΔPQR

Example 4.3

Observe Fig. 4.18 and find $\angle P$

✍ **Solution:**

In ΔBAC and ΔPRQ,



$$\frac{AB}{RQ} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{BC}{QP} = \frac{6}{12} = \frac{1}{2}; \frac{CA}{PR} = \frac{3\sqrt{3}}{6\sqrt{3}} = \frac{1}{2}$$

Therefore, $\frac{AB}{RQ} = \frac{BC}{QP} = \frac{CA}{PR}$

By SSS similarity, we have $\Delta BAC \sim \Delta QRP$

$\angle P = \angle C$ (since the corresponding parts of similar triangle)

$$\angle P = \angle C = 180° - (\angle A + \angle B)$$

$$= 180° - (90° + 60°)$$

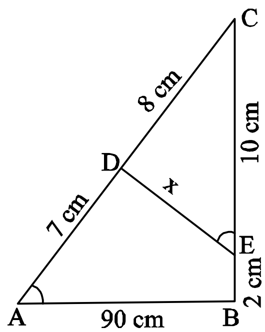
$$\angle P = 180° - 150° = 30°$$

Example 4.5

In Fig. $\angle A = \angle CED$ prove that $\Delta CAB \sim \Delta CED$. Also find the value of x.

✍ **Solution:**

In ΔCAB and ΔCED, $\angle C$ is common, $\angle A = \angle CED$



Therefore, $\Delta CAB \sim \Delta CED$ (By AA similarity)

Hence

$$\frac{CA}{CE} = \frac{AB}{DE} = \frac{CB}{CD}$$

$$\frac{AB}{DE} = \frac{CE}{CD} \text{ gives } \frac{9}{x} = \frac{10+12}{8}$$

$$\frac{9}{x} = \frac{12}{8}$$

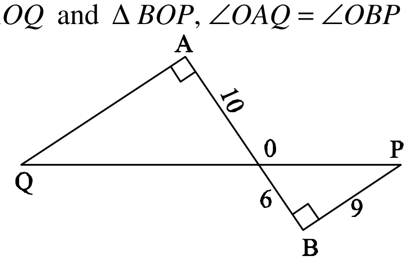
$$\text{so, } x = \frac{8 \times 9}{12} = 6 \text{ cm}$$

Example 4.6

In Fig. QA and PB are perpendicular to AB. If AO = 10 cm, BO = 6 cm and PB = 9 cm. Find AQ.

✍ **Solution:**

In ΔAOQ and ΔBOP, $\angle OAQ = \angle OBP = 90°$



$\angle AOQ = \angle BOP$ (Vertically opposite angles)

Therefore, by AA Criterion of similarity,

$\Delta AOQ \sim \Delta BOP$

$$\frac{AO}{BO} = \frac{OQ}{OP} = \frac{AQ}{BP}$$

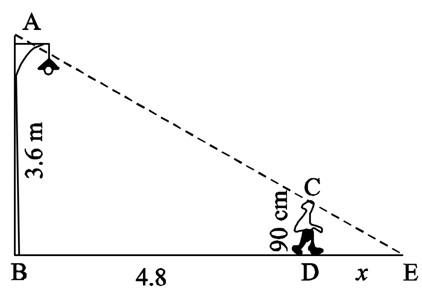
$$\frac{10}{6} = \frac{AQ}{9} \text{ gives } AQ = \frac{10 \times 9}{6} = 15 \text{ cm}$$

Example 4.4

A boy a height 90cm is walking away from the base of a lamp post at a speed of 1.2m/sec. If the lamp post is 3.6m above the ground, find the length of his shadow cast after 4 seconds.

✍ **Solution:**

Given, speed = 1.2 m/s,



time = 4 seconds

distance = speed × time

= 1.2 × 4 = 4.8 m

Let x be the length of the shadow after 4 seconds.

Since, $\Delta ABE \sim CDE$, $\frac{BE}{DE} = \frac{AB}{CD}$

$$\frac{4.8 + x}{x} = \frac{3.6}{0.9}$$

$$\frac{4.8 + x}{x} = 4$$

$$4x = 4.8 + x$$

$$4x - x = 4.8$$

$$3x = 4.8$$

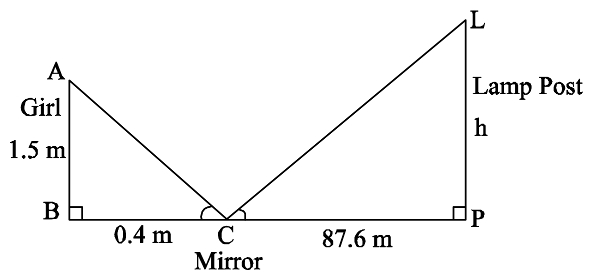
$$x = \frac{4.8}{3}$$

$$x = 1.6$$

$$\boxed{DE = 1.6 \text{ m}}$$

2. A girl looks the reflection of the top of the lamp post on the mirror which is 66 m away from the foot of the lamppost. The girl whose height is 12.5 m is standing 2.5 m away from the mirror. Assuming the mirror is placed on the ground facing the sky and the girl, mirror and the lamppost are in a same line, find the height of the lamp post.

✍ **Solution:**



Given $AB =$ height of girl = 1.5 m

$BC =$ Dist. between girl and Mirror = 0.4 m

$LP =$ height of lamp post $= h$

$CP =$ dist. between Mirror and Post = 87.6 m

In $\Delta ABC, \Delta LPC, \angle B = LP = 90°$,

$\angle ACB = \angle LCP$ (angle of incidence and angle of reflection)

$\therefore$ ΔABC and ΔLPC are similar.

$$\therefore \frac{AB}{LP} = \frac{BC}{CP} \qquad \text{(By } AA \text{ similarity)}$$

$$\Rightarrow \frac{1.5}{h} = \frac{0.4}{87.6}$$

$$\Rightarrow h = \frac{87.6 \times 1.5}{0.4}$$

$$= \frac{131.4}{0.4}$$

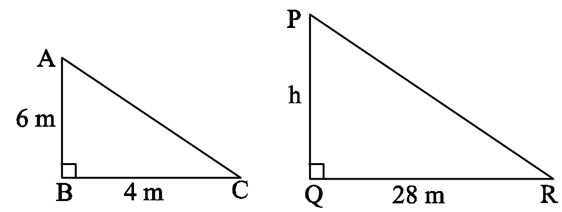
$$= \frac{1314}{4}$$

$$= 328.5$$

$\therefore$ Height of the lamp post = 328.5 m

3. A vertical stick of length 6 m casts a shadow 400 cm long on the ground and at the same time a tower casts a shadow 28 m long. Using similarity, find the height of the tower.

✍ **Solution:**



In ΔABC and ΔPQR

$$\angle B = \angle Q = 90°$$

$$\angle C = \angle R \qquad (AC \parallel PR)$$

$\therefore$ By AA similarity, $\Delta ABC \sim \Delta PQR$

$$\therefore \frac{AB}{PQ} = \frac{BC}{QR}$$

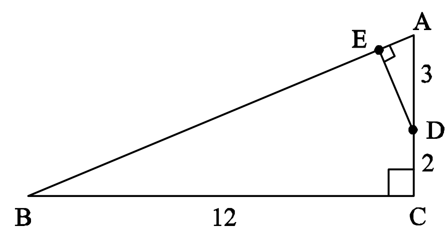
$$\Rightarrow \frac{6}{h} = \frac{4}{28}$$

$$\Rightarrow \frac{6}{h} = \frac{1}{7}$$

$$\Rightarrow h = 42 \text{ m}$$

$\therefore$ Height of the tower = 42 m.

5. In the adjacent figure, ΔABC is right angled at C and $DE \perp AB$. Prove that $\Delta ABC \sim \Delta ADE$ and hence find the lengths of AE and DE.



✍ **Solution:**

In ΔABC and ΔADE,

(i) $\angle AED = \angle ACB = 90°$

(ii) $\angle A$ is common

$\therefore$ By AA similarly,

$$\Delta ABC \sim \Delta ADE$$

Also, $AB^2 = AC^2 + BC^2$

$$= 5^2 + 12^2$$
$$= 25 + 144$$
$$= 169$$

$\therefore \quad AB = 13$

$\therefore$ By similarily,

$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE}$$

$$\frac{13}{3} = \frac{12}{DE} = \frac{5}{AE}$$

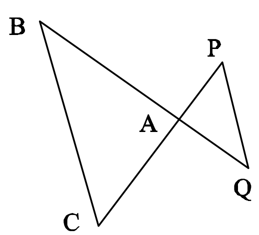
$$\frac{13}{3} = \frac{12}{DE} \qquad \frac{13}{3} = \frac{5}{AE}$$

$$13DE = 36 \qquad AE = \frac{15}{13}$$

$$DE = \frac{36}{13}$$

---

6. In the adjacent figure, $\Delta ACB \sim \Delta APQ$. If $BC = 8$ cm, $PQ = 4$ cm, $BA = 6.5$ cm and $AP = 2.8$ cm, find CA and AQ.



✍ **Solution:**

Given $\Delta ACB \sim \Delta APQ$

$$\frac{AC}{AP} = \frac{CB}{PQ} = \frac{AB}{AQ}$$

$$\frac{AC}{2.8} = \frac{8}{4} = \frac{6.5}{AQ}$$

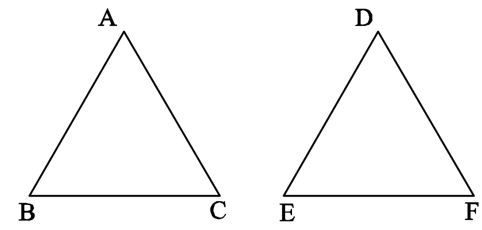
$$\frac{AC}{2.9} = \frac{8}{4}$$
$$\frac{AC}{2.8} = \frac{2}{1}$$
$$AC = 5.6 \text{ cm}$$

$$\frac{8}{4} = \frac{6.5}{AQ}$$
$$\frac{2}{1} = \frac{6.5}{AQ}$$
$$AQ = \frac{6.5}{2}$$
$$AQ = 3.25 \text{ cm}$$

---

8. If $\Delta ABC \sim \Delta DEF$ such that area of ΔABC is 9 cm^2 and the area of ΔDEF is 16 cm^2 and $BC = 2.1$ cm. Find the length of EF.

✍ **Solution:**



Given $\Delta ABC \sim \Delta DEF$

$$\therefore \quad \frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{BC^2}{EF^2}$$

$$\Rightarrow \frac{9}{16} = \frac{(2.1)^2}{EF^2}$$

$$\left(\frac{3}{4}\right)^2 = \left(\frac{2.1}{EF}\right)^2$$

$$\frac{3}{4} = \frac{2.1}{EF}$$

$$EF = \frac{201 \times 4}{3}$$

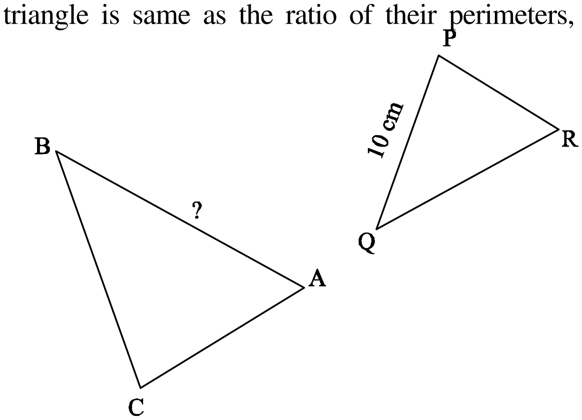
$$= 2.8 \text{ cm}$$

Example 4.7

The perimeters of two similar triangles ABC and PQR are respectively 36 cm and 24 cm. If $PQ = 10$ cm , find AB.

✍ **Solution:**

The ratio of the corresponding sides of similar triangle is same as the ratio of their perimeters,



Since $\Delta ABC \sim \Delta PQR$,

$$\frac{AB}{PQ}=\frac{BC}{QR}=\frac{AC}{PR}=\frac{36}{24}$$

$$\frac{AB}{PQ}=\frac{36}{24} \text{ gives } \frac{AB}{10}=\frac{36}{24}$$

$$AB=\frac{36\times 10}{24}=15 \text{ cm}$$

Example 4.8

If ΔABC is similar to ΔDEF such that $BC = 3$ cm, $EF = 4$ cm and area of $\Delta ABC = 54$ cm^2. Find the area of ΔDEF.

Solution:

Since the ratio of area of two similar triangles is equal to the ratio of the squares of any two corresponding sides, we have

$$\frac{\text{Area }(\Delta ABC)}{\text{Area }(\Delta DEF)}=\frac{BC^2}{EF^2} \text{ gives } \frac{54}{\text{Area }(\Delta DEF)}=\frac{3^2}{4^2}$$

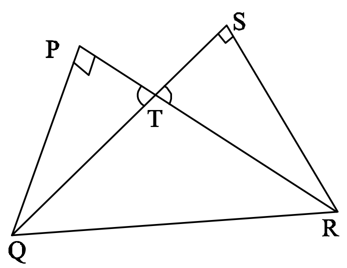
$$\text{Area }(\Delta DEF)=\frac{16\times 54}{9}=96 \text{ cm}^2$$

Type II: Prove the following based on similarity

Q.No. 4, 7, 9, Example 4.9

4. Two triangles QPR and QSR, right angled at P and S respectively are drawn on the same base QR and on the same side of QR. If PR and SQ intersect at T, prove that $PT \times TR = ST \times TQ$.

Solution:



Consider ΔPQT and ΔSRT

(i) $\angle P = \angle S = 90°$

(ii) $\angle PTQ = \angle STR$ (Vertically Opp.angle)

$\therefore$ By AA similarity,

$$\Delta PQT \sim \Delta SRT$$

$$\therefore \frac{QT}{TR}=\frac{PT}{ST}$$

$\Rightarrow PT \times TR = ST \times TQ$

Hence proved.

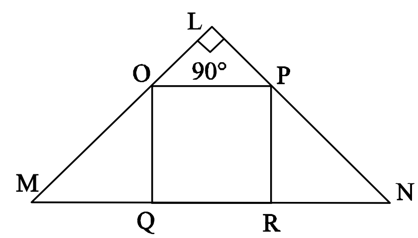
7. If figure $OPRQ$ is a square and $\angle MLN = 90°$. Prove that

(i) $\Delta LOP \sim \Delta QMO$

(ii) $\Delta LOP \sim \Delta RPN$

(iii) $\Delta QMO \sim \Delta RPN$

(iv) $QR^2 = MQ \times RN$



Solution:

(i) In $\Delta LOP, \Delta QMO$

$\angle OLP = \angle OQM = 90°$

$\angle LOP = \angle OMQ$ (Corresponding angles)

$\therefore$ By AA similarity,

$$\Delta LOP \sim \Delta QMO$$

(ii) In $\Delta LOP, \Delta RPN$

$\angle OLP = \angle PRN = 90°$

$\angle LPO = \angle PNR$ (Corresponding angles)

$\therefore$ By AA similarity,

$$\Delta LOP \sim \Delta RPN$$

$\therefore$ From (i) and (ii)

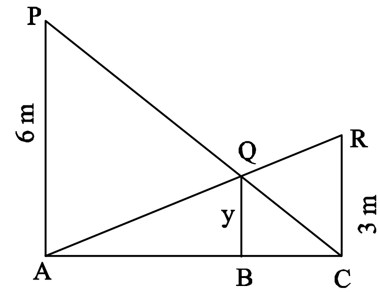
$$\Delta QMO \sim \Delta RPN$$

$$\therefore \frac{QM}{RP}=\frac{QO}{RN} \Rightarrow \frac{QM}{QR}=\frac{QR}{RN}$$

($RP = QR$) Square sides are equal.

$$\Rightarrow QR^2 = MQ \times RN$$

9. Two vertical poles of heights 6m and 3m are erected above a horizontal ground *AC*. Find the value of *y*.



Solution:

From the fig.

$\Delta\, PAC$ and $\Delta\, QBC$)

$\angle PAC = \angle QBC \;= 90°$

$\angle C$ is common

$\therefore$ by AA similarity

$\Delta\, PAC \sim QBC$

$$\therefore \frac{CB}{CA} = \frac{QB}{PA}$$

$$\Rightarrow \frac{CB}{CA} = \frac{y}{6} \quad ...(1)$$

$\Delta\, RCA$ and $\Delta\, QBA$

$\angle RCA = \angle QBA = 90°$

$\angle A$ is common

$\therefore$ by AA similarity

$\Delta\, RCA \sim \Delta\, QBA$

$$\therefore \frac{AB}{AC} = \frac{BQ}{RC} \Rightarrow \frac{AB}{BC} = \frac{y}{3} \quad ...(2)$$

Adding (1) and (2),

$$\frac{AB+BC}{AC} = \frac{y}{6} + \frac{y}{3}$$

$$\Rightarrow \frac{AC}{AC} = y\left(\frac{1}{6} + \frac{1}{3}\right)$$

$$\Rightarrow y\left(\frac{1+2}{6}\right) = 1$$

$$\Rightarrow y\left(\frac{1}{2}\right) = 1$$

$$\therefore \; y = 2m$$

(or) Using formula $y = \dfrac{ab}{a+b}$

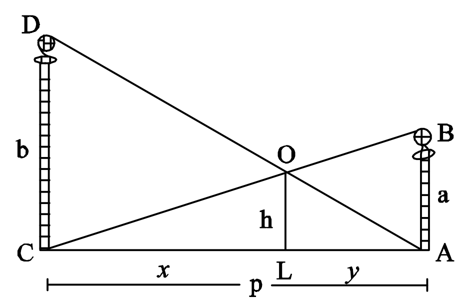
$$= \frac{6\times 3}{6+3} = \frac{18}{9} = 2\text{m}$$

Example 4.9

Two poles of height 'a' meters and 'b' meters are 'p' meters apart. Prove that the height of the point of intersection of the lines joining the top of each pole of the foot of the opposite pole is given by $\frac{ab}{a+b}$ meters.

Solution:

Let AB and CD be two poles of height 'a' meters and 'b' meters respectively such that the poles are 'p' meters apart. That is $AC = p$ meters. Suppose the lines AD and BC meet at O, such that $OL = h$ meters



Let $CL = x$ and $LA = y$.

Then, $x + y = p$

In $\Delta\, ABC$ and $\Delta\, LOC$, we have $\angle CAB = \angle CLO$ [each equals to 90°]

$\angle C = \angle C$ [C is common]

$\Delta\, CAB \sim \Delta\, CLO$ [By AA similarity]

$$\frac{CA}{CL} = \frac{AB}{LO} \text{ gives } \frac{p}{x} = \frac{a}{h}$$

so, $$x = \frac{ph}{a} \quad ...(1)$$

In $\Delta\, ALO$ and $\Delta\, ACD$, we have

$\angle ALO = \angle ACD$ [each equal to 90°]

$\angle A = \angle A$ [A is common]

Construction: Draw $CE \parallel DA$. Extend BA to meet at E.

No.	Statement	Reason
1.	$\angle BAD = \angle 1$ $\angle DAC = \angle 2$	Assumption
2.	$\angle BAD = \angle AEC$ $= \angle 1$	Since $DA \parallel CE$ and AC is transversal, corresponding angles are equal.
3.	$\angle DAC = \angle ACE$ $= \angle 2$	Since $DA \parallel CE$ and AC is transversal. Alternate angles are equals.
4.	$\frac{BA}{AE} = \frac{BD}{DC} \ldots (2)$	In ΔBCE by Thales theorem
5.	$\frac{AB}{AC} = \frac{BD}{DC}$	From (1)
6.	$\frac{AB}{AC} = \frac{BA}{AE}$	From (1) and (2)
7.	$AC = AE \ldots (3)$	Cancelling AB
8.	$\angle 1 = \angle 2$	ΔACE is isosceles by (3)
9.	AD bisects $\angle A$	Since Hence proved.

Type I: (Problems Based on Thales Theorem or BPT)

Q.No. 1(i)(ii), Example 4.12, 2, 3(i)(ii), Example 4.13, 5

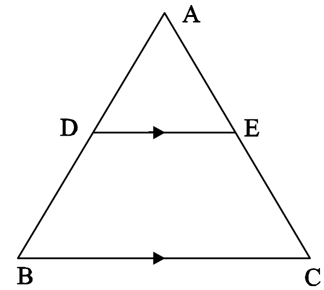
1. In $\Delta ABC, D$ and E are points on the sides AB and AC respectively such that $DE \parallel BC$

(i) If $\frac{AD}{DB} = \frac{3}{4}$ and $AC = 15$ cm find AE.

(ii) If $AD = 8x - 7, DB = 5x - 3,$ $AE = 4x - 3$ and $EC = 3x - 1$, find the value of x.

✍ **Solution:**

(i) Given $\frac{AD}{DB} = -\frac{3}{4}, AC = 15$



Let $AE = x \Rightarrow EC = 15 - x$

$\therefore$ By $BPT \ \frac{AD}{DB} = \frac{AE}{EC}$

$\Rightarrow \frac{3}{4} = \frac{x}{15 - x}$

$\Rightarrow 4x = 45 - 3x$

$\Rightarrow 7x = 45$

$x = \frac{45}{7} = 6.428$

$= 6.43$

(ii) Given $AD = 8x - 7, DB = 5x - 3$

$AE = 4x - 3, EC = 3x - 1$

By $BPT \ \frac{AD}{DB} = \frac{AE}{EC}$

$\Rightarrow \frac{8x - 7}{5x - 3} = \frac{4x - 3}{3x - 1}$

$\Rightarrow (8x - 7)(3x - 1) = (4x - 3)(5x - 3)$

$24x^2 - 8x - 21x + 7 = 20x^2 - 12x - 15x + 9$

$\Rightarrow 24x^2 - 29x + 7 = 20x^2 - 27x + 9$

$\Rightarrow 4x^2 - 2x - 0 = 0$

$\Rightarrow 2x^2 - x - 1 = 0$

$\therefore x = 1, \frac{-1}{2}$

(negative not possible)

$\therefore x = 1$ only

-2

$-2 \quad 1$

$\frac{-2}{2}, \quad \frac{1}{2}$

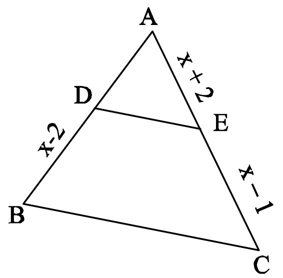
$-1, \quad \frac{1}{2}$

Example 4.12

In ΔABC if $DE || BC, AD = x, DB = x - 2,$ and $EC = x - 1$ then find the lengths of the sides AB and AC.

✍ **Solution:**

In ΔABC we have $DE || BC$



By Thales theorem, we have $\frac{AD}{DB} = \frac{AE}{EC}$

$\frac{x}{x-2} = \frac{x+2}{x-1}$ gives $x(x-1) = (x-2)(x+2)$

Hence, $x^2 - x = x^2 - 4$ so, $x = 4$

When $x = 4, AD = 4, DB = x - 2 = 2,$

$AE = x + 2 = 6, EC = x - 1 = 3$

Hence, $AB = AD + DB = 4 + 2 = 6$

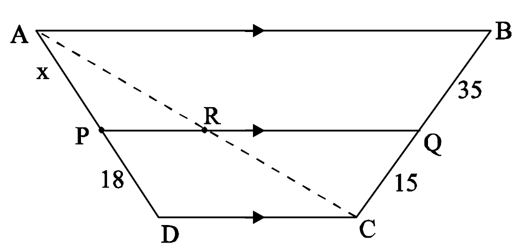
$AC = AE + EC = 6 + 3 = 9$

Therefore, $AB = 6, AC = 9$

---

2. $ABCD$ is a trapezium in which $AB \parallel DC$ and P, Q are points on AD and BC respectively, such that $PQ \parallel DC$ if $PD = 18$ cm, $BQ = 35$ cm and $QC = 15$ cm, find AD

✍ **Solution:**



In trapezium $ABCD, AB \parallel DC \parallel PQ$

Join AC, meet PQ at R.

In $\Delta ACD, PR \parallel DC$

$\therefore$ By $BPT \frac{AP}{PD} = \frac{AR}{RC}$

$\Rightarrow \frac{x}{18} = \frac{AR}{RC}$ ...(1)

In $\Delta ABC, RQ \parallel AB$

$\therefore$ By $ABT \frac{BQ}{QC} = \frac{AR}{RC}$

$\Rightarrow \frac{35}{15} = \frac{AR}{RC}$

$\Rightarrow \frac{7}{3} = \frac{AR}{RC}$ ...(2)

$\therefore$ From (1) and (2)

$\frac{x}{18} = \frac{7}{3}$

$\frac{x}{6} = \frac{7}{1}$

$\Rightarrow x = 42$

$\therefore AD = AP + PD$

$= 42 + 18$

$= 60$ m

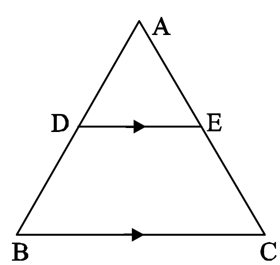
---

3. In $\Delta ABC, D$ and E are points on the sides AB and AC respectively. For each of the following cases show that $DE \parallel BC$

(i) $AB = 12$ cm, $AD = 8$ cm, $AE = 12$ cm and $AC = 18$ cm

(ii) $AB = 5.6$ cm, $AD = 1.4$ cm, $AC = 7.2$ cm and $AE = 1.8$ cm

✍ **Solution:**



In ΔABC, To Prove: $DE \parallel BC$

(i) $\frac{AD}{AB} = \frac{8}{12} = \frac{2}{3}$

$\frac{AE}{AC} = \frac{12}{18} = \frac{2}{3}$

$\frac{AD}{AB} = \frac{AE}{AC}$

$\therefore$ By converse of $BPT \; DE \parallel BC$

(ii) $\frac{AD}{AB} = \frac{1.4}{5.6} = \frac{1}{4}$

$\frac{AE}{AC} = \frac{1.8}{7.2} = \frac{1}{4}$

$\therefore \frac{AD}{AB} = \frac{AE}{AC}$

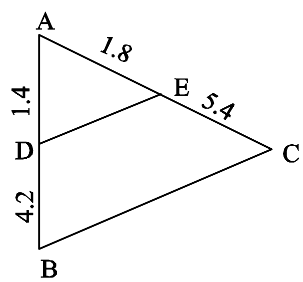
$\therefore$ By Converse of BPT, $DE \parallel BC$

Example 4.13

D and E are respectively the points on the sides AB and AC of a ΔABC such that $AB = 5.6$ cm, $AD = 1.4$ cm, $AC = 7.2$ cm AND $AE = 1.8$ cm show that $DE \parallel BC$.

Solution:

We have $AB = 5.6$ cm, $AD = 1.4$ cm, $AC = 7.2$ cm and $AE = 1.8$ cm



$BD = AB - AD = 5.6 - 1.4 = 4.2$ cm

and $EC = AC - AE = 7.2 - 1.8 = 5.4$ cm

$\frac{AD}{DB} = \frac{1.4}{4.2} = \frac{1}{3}$ and $\frac{AE}{EC} = \frac{1.8}{5.4} = \frac{1}{3}$

$\frac{AD}{DB} = \frac{AE}{EC}$

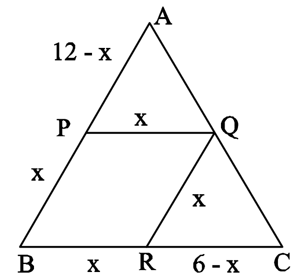
Therefore, by converse of Basic Proportionality Theorem, we have DE is parallel to BC.

Hence proved.

5. Rhombus $PQRB$ is inscribed in ΔABC such that $\angle B$ is one of its angle. P, Q and R lie on AB, AC and BC respectively. If $AB = 12$ cm and $BC = 6$ cm, find the sides PQ, RB of the rhombus.

Solution:

Rhombus $PQRS$ is inscribed in ΔABC



Let the side of the rhombus be x

$\therefore AB = 12$ cm $AP = 12 - x$

$BC = 6$ cm $RC = 6 - x$

In ΔABC, $PQ \parallel BC$

$$\therefore \frac{AP}{PB} = \frac{AQ}{QC} \quad ...(1)$$

In ΔABC, $QR \parallel AB$

$$\therefore \frac{BR}{RC} = \frac{AQ}{QC} \quad ...(2)$$

$\therefore$ From (1) and (2)

$$\Rightarrow \frac{AP}{PB} = \frac{BR}{RC}$$

$$\Rightarrow \frac{12 - x}{x} = \frac{x}{6 - x}$$

$$\Rightarrow x^2 = (6 - x)(12 - x)$$

$$\Rightarrow x^2 = x^2 - 18x + 72$$

$$\Rightarrow 18x = 72$$

$$\Rightarrow x = 4 \text{ cm}$$

$$\therefore PQ = RB = 4 \text{ cm}$$

Type II: (Prove the followings based on BPT)

Q.No. 4, 6, 7, Example 4.14

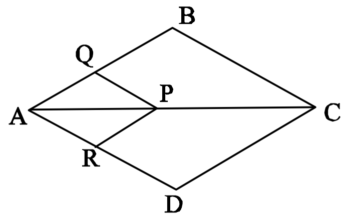
4. In fig. $PQ \parallel BC$ and $PR \parallel CD$ prove that

(i) $\frac{AR}{AD} = \frac{AQ}{AB}$ (ii) $\frac{QB}{AQ} = \frac{DR}{AR}$

Solution:

(i) In ΔABC, $PQ \parallel BC$

$$\therefore \text{By } BPT \frac{AQ}{AB} = \frac{AP}{AC} \quad ...(1)$$



In ΔADC, $PR \parallel DC$

$\therefore$ By BPT $\dfrac{AR}{AD} = \dfrac{AP}{AC}$ ...(2)

$\therefore$ From (1) and (2),

$$\frac{AQ}{AB} = \frac{AR}{AD}$$

(ii) From (i) $\dfrac{AB}{AQ} = \dfrac{AD}{AR}$ (reciprocal)

$$\Rightarrow \frac{AB}{AQ} - 1 = \frac{AD}{AR} - 1$$

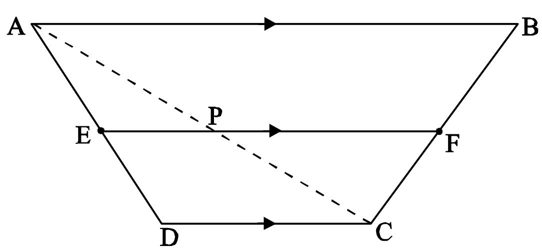
$$\Rightarrow \frac{AB - AQ}{AQ} = \frac{AD - AR}{AR}$$

$$\Rightarrow \frac{BQ}{AQ} = \frac{DR}{AR}$$

Hence proved.

6. In trapezium $ABCD$, $AB \parallel DC$, E and F are points on non-parallel sides AD and BC respectively, such that $EF \parallel AB$. Show that $\dfrac{AE}{ED} = \dfrac{BF}{FC}$

✍ **Solution:**



In trapezium $ABCD$, $AB \parallel DC \parallel EF$

Join AC to meet EF at P

In ΔADC, $EP \parallel DC$

$\therefore$ By BPT, $\dfrac{AE}{ED} = \dfrac{AP}{PC}$ ...(1)

In ΔABC, $PR \parallel AB$

$\therefore$ By BPT, $\dfrac{BF}{FC} = \dfrac{AP}{PC}$ ...(2)

From (1) and (2)

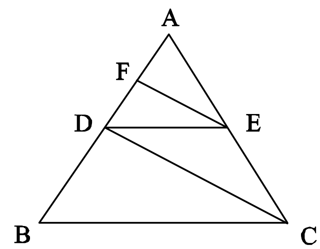
$$\frac{AE}{ED} = \frac{BF}{FC}$$

Hence proved.

7. In figure $DE \parallel BC$ and $CD \parallel EF$. Prove that $AD^2 = AB \times AF$

✍ **Solution:**

In figure $DE \parallel BC$ and $CD \parallel EF$



In ΔACD, by BPT, $\dfrac{AF}{AD} = \dfrac{AE}{AC}$ ...(1)

In ΔABC, by BPT, $\dfrac{AD}{AB} = \dfrac{AE}{AC}$ ...(2)

$\therefore$ From (1) and (2)

$$\frac{AF}{AD} = \frac{AD}{AB}$$

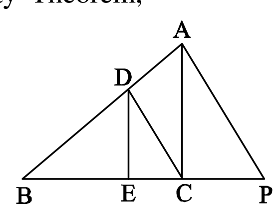
$$\Rightarrow AD^2 = AF \cdot AB$$

Example 4.14

In the Fig. $DE \parallel AC$ and $DC \parallel AP$. Prove that $\dfrac{BE}{EC} = \dfrac{BC}{CP}$.

✍ **Solution:**

In ΔBPA, we have $DC \parallel AP$. By Basic Proportionality Theorem,



We have $\frac{BC}{CP} = \frac{BD}{DA}$ ...(1)

In ΔBCA, we have $DE \parallel AC$. By Basic Proportionality Theorem,

We have $\frac{BE}{EC} = \frac{BD}{DA}$ ...(2)

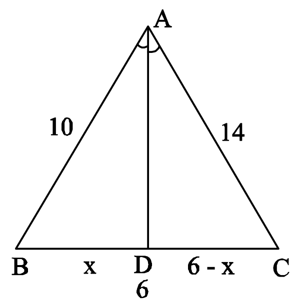
From (1) and (2) we get, $\frac{BE}{EC} = \frac{BC}{CP}$.

Hence proved.

Type III: (Problems based on ABT)
Q.No. 8, Example 4.15, 4.16, 9(i)(ii)

8. In $\Delta ABC, AD$ is the bisector of $\angle A$ meeting side BC at D, if $AB = 10$ cm, $AC = 14$ cm and $BC = 6$ cm, find BD and DC

✍ **Solution:**



In $\Delta ABC, AD$ is the bisector of $\angle A$

$\therefore$ By $ABT, \frac{AB}{AC} = \frac{BD}{DC}$

$$\Rightarrow \frac{10}{14} = \frac{x}{6-x}$$

$$\Rightarrow \frac{5}{7} = \frac{x}{6-x}$$

$$\Rightarrow 30 - 5x = 7x$$

$$\Rightarrow 12x = 30$$

$x = \frac{5}{2} = 2.5$

$\therefore$ $BD = 2.5$ cm and $DC = 6 - x = 6 - 2.5$

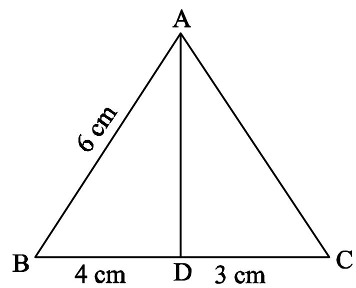
$DC = 3.5$ cm

Example 4.15

In the Fig., AD is the bisector of $\angle A$. If $BD = 4$ cm, $DC = 3$ cm and $AB = 6$ cm, find AC.

✍ **Solution:**

In $\Delta ABC, AD$ is the bisector of $\angle A$



Therefore by Angle Bisector Theorem

$$\frac{BD}{DC} = \frac{AB}{AC}$$

$$\frac{4}{3} = \frac{6}{AC} \text{ gives } 4\,AC = 18$$

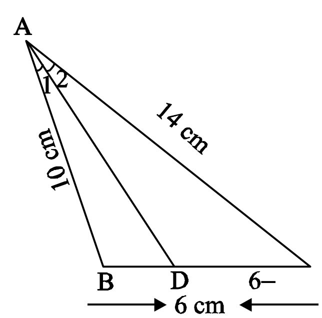
Hence $AC = \frac{9}{2} = 4.5$ cm

Example 4.16

In the Fig. AD is the bisector of $\angle BAC$. If $AB = 10$ cm, $AC = 14$ cm and $BC = 6$ cm, find BD and DC.

✍ **Solution:**

Let $BD = x$ cm, then $DC = (6 - x)$ cm



AD is the bisector of $\angle A$

Therefore by Angle Bisector Theorem

$$\frac{AB}{AC} = \frac{BD}{DC}$$

$$\frac{10}{14} = \frac{x}{6-x} \text{ gives } \frac{5}{7} = \frac{x}{6-x}$$

So, $12x = 30$ we get, $x = \frac{30}{12} = 2.5$ cm

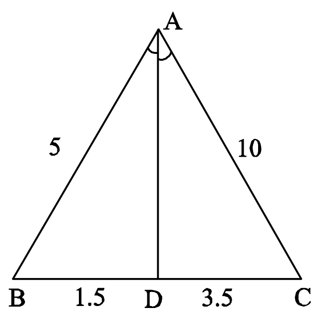
Therefore, $BD = 2.5$ cm, $DC = 6 - x$

$= 6 - 2.5 = 3.5$ cm

9. Check whether AD is bisector of $\angle A$ of ΔABC in each of the following
(i) $AB = 5$ cm, $AC = 10$ cm, $BD = 1.5$ cm and $CD = 3.5$ cm.
(ii) $AB = 4$ cm, $AC = 6$ cm, $BD = 1.6.$ cm and $CD = 2.4$ cm

✍ **Solution:**

(i)

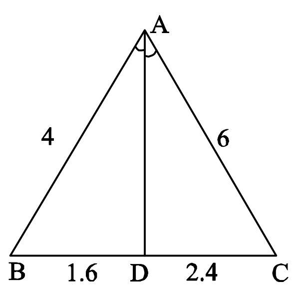


$$\frac{AB}{AC} = \frac{5}{10} = \frac{1}{2}, \frac{BD}{DC} = \frac{1.5}{3.5} = \frac{3}{7}$$

$$\therefore \frac{AB}{AC} \neq \frac{BD}{DC}$$

$\therefore AD$ is not the bisector of $\angle A$.

(ii)



$$\frac{AB}{AC} = \frac{4}{6} = \frac{2}{3}, \frac{BD}{DC} = \frac{1.6}{2.4} = \frac{2}{3}$$

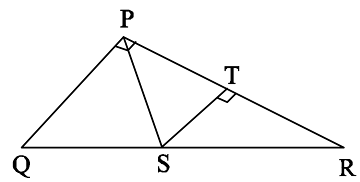
$$\therefore \frac{AB}{AC} = \frac{BD}{DC}$$

$\therefore$ By Converse of ABT,

AD is the bisector of $\angle A$

Type IV: (Prove the following based on ABT)
Q.No. 10, 11

10. In figure $\angle QPR = 90°$, PS is its bisector if $ST \perp PR$. Prove that $ST \times (PQ + PR) = PQ \times PR$



Here $ST = PT$

Area of ΔPQR = Area of ❑ $STPQ$ + Area of ΔSTR

$$\frac{1}{2} PQ \cdot PR = \frac{1}{2} [PT\,(ST + PQ) + \frac{1}{2} (ST \cdot TR)]$$

$$PQ \cdot PR = PT \cdot ST + PT \cdot PQ + ST \cdot TR$$

$$= ST\,(PT + TR) + PT \cdot PQ$$

$$= ST \cdot PR + PT \cdot PQ$$

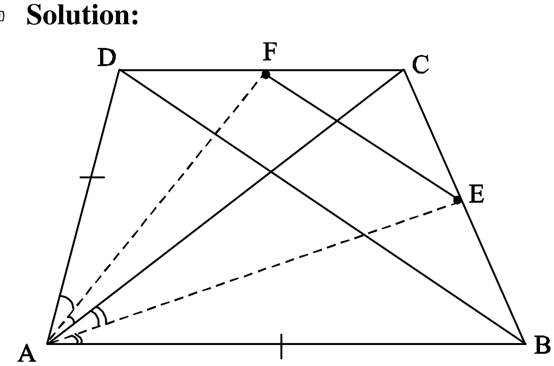
$$= ST \cdot PR + ST \cdot PQ$$

$$= ST\,(PR + PQ)$$

Hence proved.

11. $ABCD$ is a quadrilateral in which $AB = AD$, the bisector of $\angle BAC$ and $\angle CAD$ intersect the sides BC and CD at the points E and F respectively. Prove that $EF \parallel BD$.

✍ **Solution:**



In $\Delta\, ACD$, AF is the angle bisector

$\therefore$ By ABT, $\dfrac{AD}{AC} = \dfrac{DF}{FC}$ ...(1)

In $\Delta\, ABC$, AE is the angle bisector

$\therefore$ By ABT, $\dfrac{AB}{AC} = \dfrac{BE}{EC}$

$\Rightarrow \dfrac{AD}{AC} = \dfrac{BE}{EC}$ ...(2) (Given $AB = AD$)

$\therefore$ From (1) and (2),

$$\frac{BE}{EC} = \frac{DF}{FC}$$

$\therefore$ By Converse of BPT,

$$EF \parallel BD$$

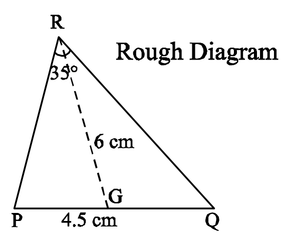
Hence proved.

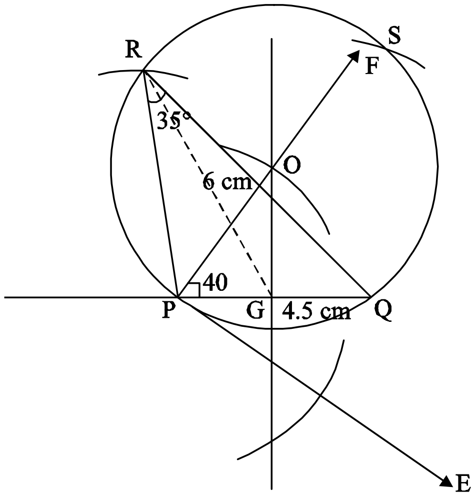
Construction of Triangle

Type I. Given base, vertical angle and median
Q.No. 12, 13, Example 4.17, 4.18

12. Construct a $\Delta\, PQR$ which the base $PQ = 4.5$ cm, $\angle R = 35°$ and the median from R to PQ is 6 cm.

✍ **Solution:**





Construction

Step 1: Draw a line segment $PQ = 4.5$ cm

Step 2: At P, draw PE such that $\angle QPE = 35°$.

Step 3: At P, draw PE such that $\angle EPF = 90°$

Step 4: Draw the perpendicular bisector to PQ, meets PF at O and PQ at G.

Step 5: With O as centre and OP as radius draw a circle.

Step 6: From G mark arcs of 6 cm on the circle at RAS.

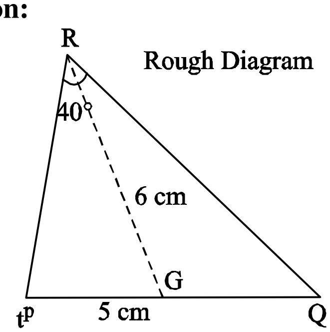
Step 7: Join PR, RQ. Then $\Delta\, PQR$ is the required Δ.

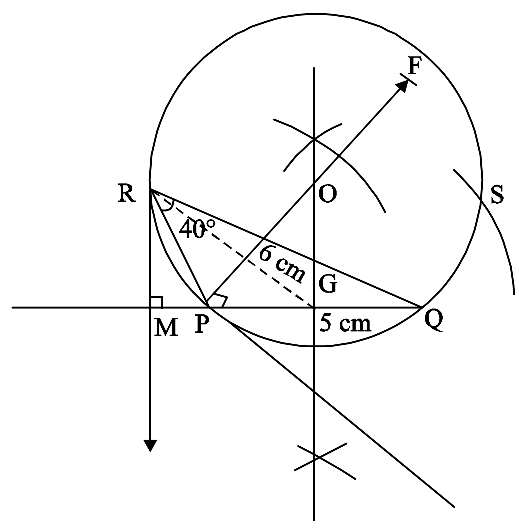
Step 8: Join RG, which is the median.

---

13. Construct a $\Delta\, PQR$ in which $PQ = 5$ cm, $\angle P = 40°$ and the median PG from P to QR is 4.4 cm. Find the length of the altitude from P to QR.

✍ **Solution:**





Construction

Step 1: Draw a line segment $PQ = 5$ cm

Step 2: At P, draw PE such that $\angle QPE = 40°$

Step 3: At P, draw PF such that $\angle EPF = 90°$

Step 4: Draw the perpendicular bisector to PQ, meets PF at O and PQ at G.

Step 5: With O as centre and OP as radius draw a circle.

Step 6: From G mark arc of 4.4 cm on the circle radius 4.4 m.

Step 7: Join PR, RQ. Then ΔPQR is the required Δ.

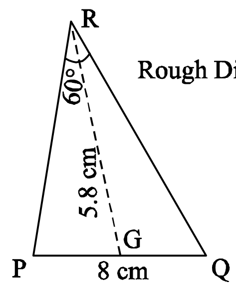
Step 8: Length of altitude is $RM = 3$ cm

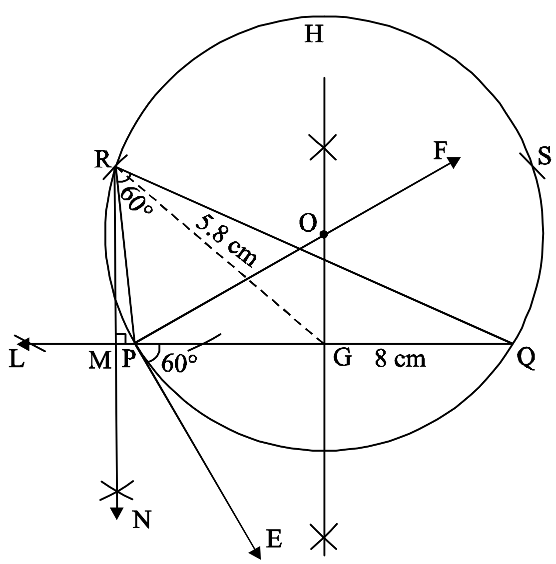
Example 4.17

Construct a ΔPQR in which $PQ = 8$ cm, $\angle R = 60°$ and the median RG from R to PQ is 5.8 cm. Find the length of the altitude from R to PQ.

✍ **Solution:**

Rough Diagram





Construction

Step 1: Draw a line segment $PQ = 8$ cm

Step 2: At P, draw PE such that $\angle QPE = 60°$

Step 3: At P, draw PF such that $\angle EPF = 90°$.

Step 4: Draw the perpendicular bisector to PQ, which intersects PF at O and PQ at G.

Step 5: With O as centre and OP as radius draw a circle.

Step 6: From G mark arcs of radius 5.8 cm on the circle. Mark them as R and S.

Step 7: Join PR and RQ. Then ΔPQR is the required triangle.

Step 8: From R draw a line RN perpendicular to LQ. LQ meets RN at M

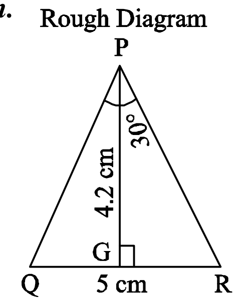
Step 9: The length of the altitude is $RM = 3.5$ cm

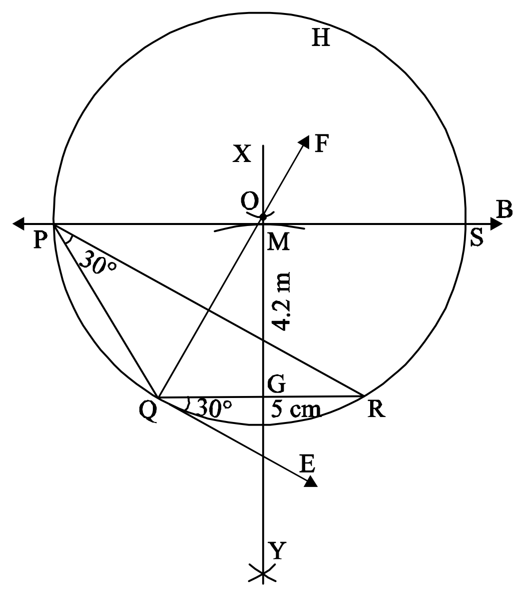
Example 4.18

Construct a triangle ΔPQR such that $QR = 5$ cm, $\angle P = 30°$ and the altitude from P to QR is of length 4.2 cm.

✍ **Solution:**

Rough Diagram





Construction

Step 1: Draw a line segment $QR = 5$ cm

Step 2: At Q, draw QE such that $\angle RQE = 30°$

Step 3: At Q, draw QF such that $\angle EQF = 90°$.

Step 4: Draw the perpendicular bisector XY to QR, which intersects QF at O and QR at G.

Step 5: With O as centre and OQ as radius draw a circle.

Step 6: From G mark an arc in the line XY at M, such that $GM = 4.2$ cm

Step 7: Draw AB through M which is parallel to QR.

Step 8: AB meets the circle at P and S

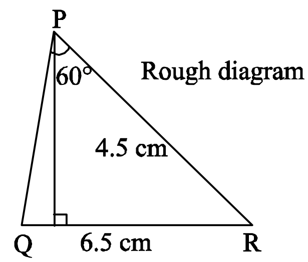
Step 9: Join QP and RP. Then ΔPQR is the required triangle.

Type II. Given base, vertical angle and altitude

Q.No. 14, 15

14. Construct a ΔPQR such that $QR = 6.5$ cm, $\angle P = 60°$ and the altitude from P to QR is of length 4.5 cm.

✍ **Solution:**



Construction

Step 1: Draw a line segment $QR = 6.5$ cm

Step 2: At Q, draw QE such that $\angle RQE = 60°$

Step 3: At Q, draw QF such that $\angle EQF = 90°$

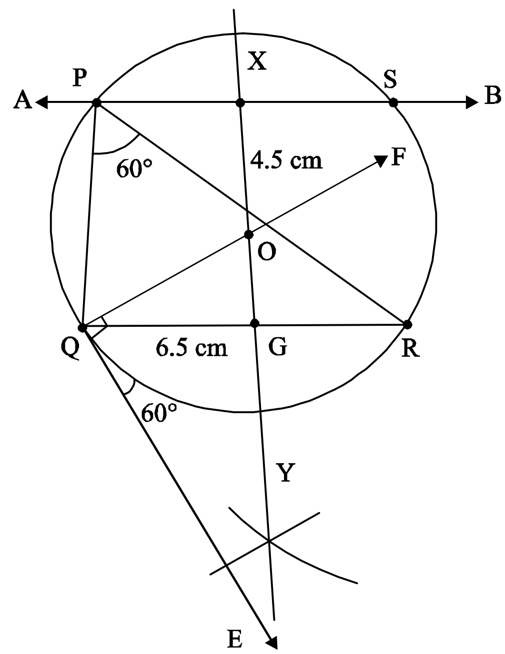
Step 4: Draw the perpendicular bisector XY to QR intersects QF at O and QR at G.

Step 5: With O as centre and OQ as radius draw a circle.

Step 6: XY intersects QR at G. On XY, from G, mark arc M such that $GM = 4.5$ cm.

Step 7: Draw AB, through M which is parallel to QR.

Step 8: AB meets the circle at P and S.



Step 9: Join QP, RP. Then ΔPQR is the required Δ.

---

15. Construct a ΔABC such that $AB = 5.5$ cm, $\angle C = 25°$ and the altitude from C to AB is 4 cm.

✍ **Solution:**

Construction

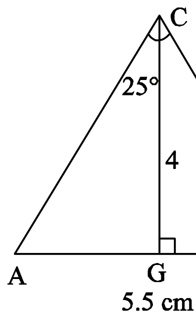
Step 1: Draw a line segment $AB = 5.5$ cm

Step 2: At A, draw AE such that $\angle BAF = 25°$

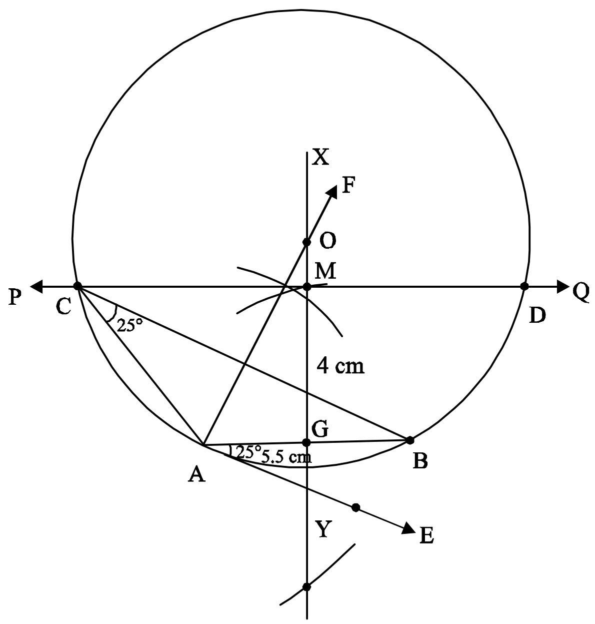
Step 3: At A, draw AF such that $\angle EAF = 90°$

Step 4: Draw the perpendicular bisector XY to AB intersects AF at O and AB at G.

Step 5: With O as centre and OA as radius draw a circle.



Rough diagram



Step 6: XY intersects AB at G. On XY, from G, mark arc M such that $GM = 4$ cm

Step 7: Draw PQ, through M parallel to AB meets the circle at C and D.

Step 8: Join AC, BC. Then ΔABC is the required Δ.

Type III. Given base, vertical angle and bisector of the vertical angle

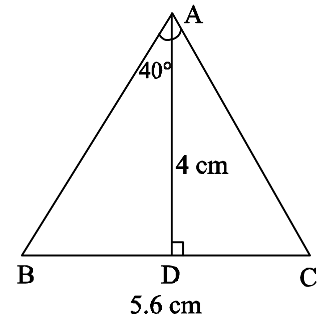
Q.No. 16, 17, Example 4.19

16. Draw a triangle ABC of base $BC = 5.6$ cm, $\angle A = 40°$ and the bisector of $\angle A$ meets BC at D such that $CD = 4$ cm

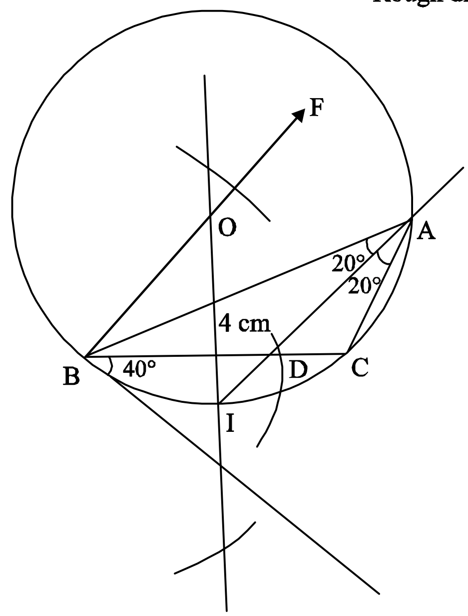
✍ **Solution:**

Construction

Step 1: Draw a line segment $BC = 5.6$ cm



Rough diagram



Step 2: At B, draw BE such that $\angle CBE = 40°$

Step 3: At B, draw BF such that $\angle CBF = 90°$

Step 4: Draw the perpendicular bisector to BC meets BF at O and BC at G

Step 5: With O as centre and OB as radius draw a circle.

Step 6: From B, mark an arc of 4 cm on BC at D.

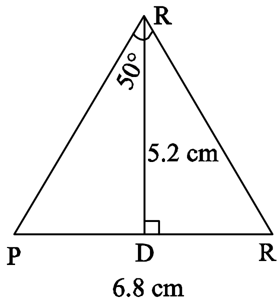
Step 7: The $\perp r$ bisector meets the circle at I and Join ID.

Step 8: ID produced meets the circle at A. Join AB and AC.

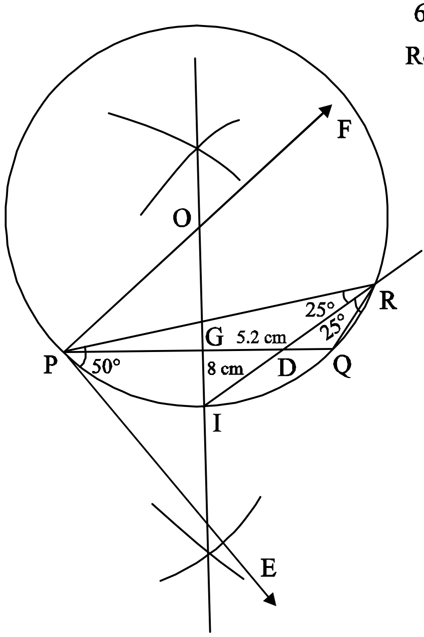
Step 9: Then ΔABC is the required triangle.

17. Draw ΔPQR such that $PQ = 6.8$ cm, vertical angle is $50°$ and the bisector of the vertical angle meets the base at D where $PD = 5.2$ cm

✍ **Solution:**



Rough diagram



Construction

Step 1: Draw a line segment $PQ = 6.8$ cm

Step 2: At P, draw PE such that $\angle QPE = 50°$

Step 3: At P, draw PF such that $\angle QPF = 90°$

Step 4: Draw the perpendicular bisector to PQ meets PF at O and PQ at G.

Step 5: With O as centre and OP as radius draw a circle.

Step 6: From P mark an arc of 5.2 cm on PQ at D.

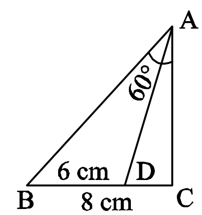
Step 7: The perpendicular bisector meets the circle at R. Join PR and QR.

Step 8: Then ΔPQR is the required triangle.

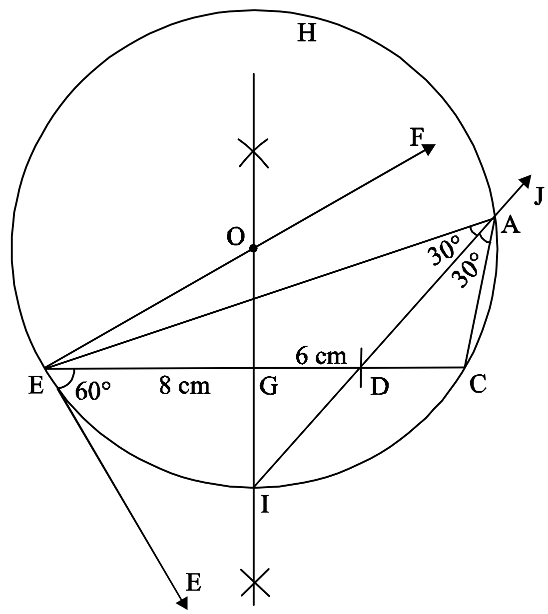
Example 4.19

Draw a triangle ABC of base $BC = 8$ cm, $\angle A = 60°$ and the bisector of $\angle A$ meets BC at D such that $BD = 6$ cm

✍ **Solution:**



Rough Diagram



Construction

Step 1: Draw a line segment $BC = 8$ cm

Step 2: At B, draw BE such that $\angle CBE = 60°$

Step 3: At B, draw BF such that $\angle EBF = 90°$

Step 4: Draw the perpendicular bisector to BC, which intersects BF at O and BC at G.

Step 5: With O as centre and OB as radius draw a circle.

Step 6: From B mark an arcs of 6 cm on BC at D.

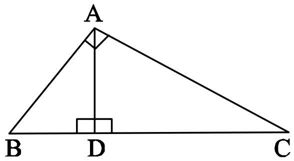
Step 7: The perpendicular bisector intersects the circle at I. Join ID.

Step 8: ID produced meets the circle at A. Now join AB and AC. Then ΔABC is the required triangle.

1. Pythagoras Theorem

Theorem 5: Pythagoras Theorem

Statement



In a right angle triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Proof

Given: In ΔABC, $\angle A = 90°$

To prove: $AB^2 + AC^2 = BC^2$

Construction: Draw $AD \perp BC$

No.	Statement	Reason
1.	Compare ΔABC and ΔABD $\angle B$ is common $\angle BAC = \angle BDA$ $= 90°$ Therefore, $\Delta ABC \sim \Delta ABD$ $\frac{AB}{BD} = \frac{BC}{AB}$ $AB^2 = BC \times BD$ ...(1)	Given $\angle BAC = 90°$ and by construction $\angle BDA = 90°$ By AA similarity
2.	Compare ΔABC and ΔADC $\angle C$ is common $\angle BAC = \angle ADC$ $= 90°$ Therefore $\Delta ABC \sim \Delta ADC$ $\frac{BC}{AC} = \frac{AC}{DC}$ $AC^2 = BC \times DC$ ...(2)	Given $\angle BAC = 90°$ and by construction $\angle CDA = 90°$ By AA similarity
3.	$AB^2 + AC^2$ $BC \times BD + BC \times DC$ $= BC(BD + DC)$ $= BC \times BC$ $= BC^2$	Add (1) and (2)

Converse of Pythagoras Theorem

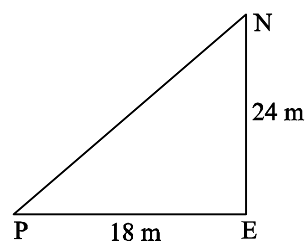
Statement

If the square of the longest side of a triangle is equal to sum of squares of other two sides, then the triangle is a right angle triangle.

Exercise 4.3

Type I: (Problems based on Pythagoras Theorem)

Q.No. 1, 2, 3, 4, 5, 6, Example 4.20, 4.21, 4.22, 4.23



1. A man goes 18m due east and then 24 m due north. Find the distance of his current position from the starting point?

✍ **Solution:**

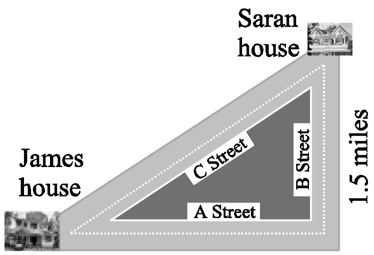
$P \rightarrow$ Starting Point

By Pythagoras Theorem,

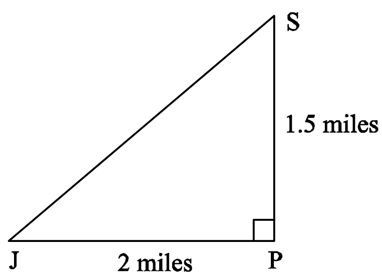
$$PN = \sqrt{18^2 + 24^2}$$
$$= \sqrt{324 + 576}$$
$$= \sqrt{900}$$
$$= 30 \text{ m}$$

$\therefore$ Distance of his current position from the starting point = 30 m

2. There are two paths that one can choose to go from Sarah's house to James house. One way is to take *C* street, and the other way requires to take *A* street and then *B* street. How much shorter is the direct path along *C* street? (Using figure)



✍ **Solution:**



Path - 1 (Direct *C* Street)

$$SJ = \sqrt{(1.5)^2 + 2^2}$$

$$= \sqrt{2.25 + 4}$$

$$= \sqrt{6.25}$$

$$= 2.5 \text{ miles}$$

Path = 2 (*B* Street and then *A* Street)

$$SP + PJ = 1.5 + 2$$

$$= 3.5 \text{ miles}$$

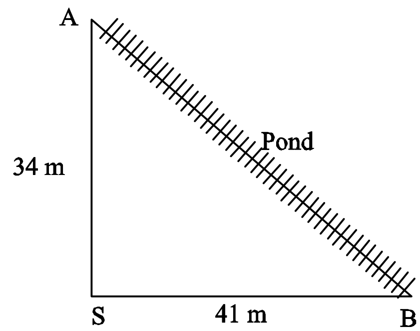
$\therefore$ Required $= 3.5 - 2.5$

$= 1$ mile

$\therefore$ 1 mile is shorter along *C* Street.

3. To get from point *A* to point *B* you must avoid walking through a pond. You must walk 34 m south and 41 m east. To the nearest meter, how many meters would be saved if it were possible to make a way through the pond?

✍ **Solution:**



Path - 1 (Through pond)

$$AB = \sqrt{34^2 + 41^2}$$

$$= \sqrt{1156 + 1681}$$

$$= \sqrt{2837}$$

$$= 53.26 \text{ m}$$

Path - 2 (South and then East)

Total dist. covered

$$AB = AS + SB$$

$$= 34 + 41$$

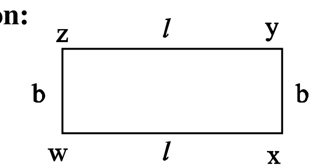
$$= 75 \text{ m}$$

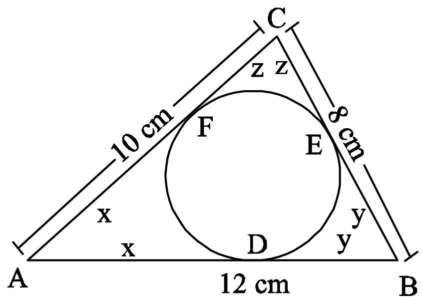
$\therefore$ Reqd. time saving $= 75 - 53.26$

$= 21.74$ m

4. In the rectangle $WXYZ$, $XY + YZ = 17$ cm, and $XZ + YW = 26$ cm. Calculate the length and breadth of the rectangle.

✍ **Solution:**





$\Rightarrow \quad x + y + z = 15$

$\Rightarrow \quad 12 + z = 15$

$\therefore \quad z = 3$

$\Rightarrow \quad y + 3 = 8$

$\Rightarrow \quad y = 5$

$\therefore \quad x + 5 = 12$

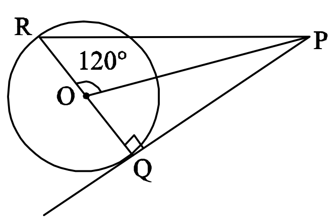
$\Rightarrow \quad x = 7$

$\therefore \quad AD = 7$ cm, $BE = 5$ cm, $CF = 3$ cm

---

4. *PQ* is a tangent drawn from a point *P* to a circle with centre *O* and *QOR* is a diameter of the circle such that $\angle PQR = 120°$. Find $\angle OPQ$

✍ **Solution:**



Given $\angle POR = 120°$

$\Rightarrow \quad \angle POQ = 60°$ (linear pair)

Also $\angle OQP = 90°$ (Radius $\perp$ tangent)

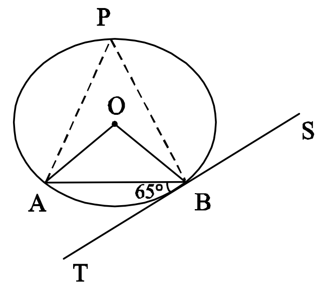
$\therefore \quad \angle OPQ = 90° - 60°$

$= 30°$

---

5. A tangent *ST* to a circle touches it at *B*. *AB* is a chord such that $\angle ABT = 65°$. Find $\angle AOB$, where "*O*" is the centre of the circle.

✍ **Solution:**



Given $\angle TBA = 65° \Rightarrow \angle APB = 65°$

(angles in ultimate segment).

$\therefore \quad \angle AOB = 2\,\angle APB = 2\,(65°) = 130°$

circumference)

(Angle substantiate at the centre is twice the angle subtended at any point on the remaining.

---

6. In figure, *O* is the centre of the circle with radius 5 cm. *T* is a point such that $OT = 13$ cm and *OT* intersects the circle *E*, if *AB* is the tangent to the circle at *E*, find the length of *AB*.

✍ **Solution:In the figure, given**

$OP = 5, OT = 13$

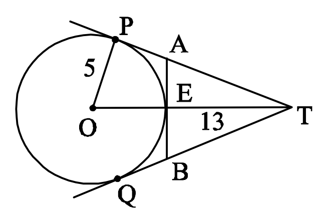
$\therefore \quad PT = \sqrt{13^2 - 5^2}$

$= \sqrt{169 - 25}$

$= \sqrt{144}$

$= 12$

$= TQ$



Also, $OE = 5 \Rightarrow ET = 13 - 5 = 8$

Let $AP = AE = x \Rightarrow TA = 12 - x$

$\therefore$ In $\Delta\, AET$, $\angle AET = 90°$

$\therefore \quad x^2 + 8^2 = (12 - x)^2$

$\Rightarrow \quad x^2 + 8^2 = 144 + x^2 - 24x$

$$\Rightarrow \quad 64 = 144 - 24x$$

$$\Rightarrow \quad 24x = 80$$

$$x = \frac{80}{24}$$

$$x = \frac{10}{3}$$

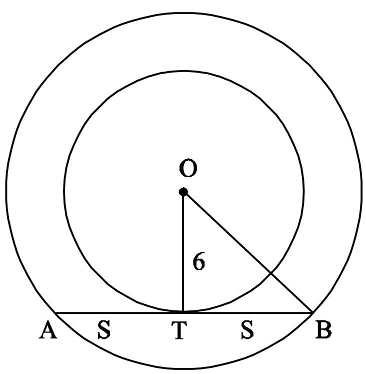
$$\therefore \quad AB = 2x$$

Length of tangent $AB = \frac{20}{3}$ cm

---

7. In two concentric circles, a chord of length 16 cm of larger circle becomes a tangent to the smaller circle whose radius is 6 cm. Find the radius of the larger circle.

✍ **Solution:**



Given the chord AB of larger circle is a tangent for the smaller circle and OT is radius.

OT is perpendicular to AB.

$\therefore \quad AT = TB = 8$ cm, $OT = 6$ cm

$\therefore$ In $\Delta\, OBT$,

$$OB = \sqrt{8^2 + 6^2}$$

$$= \sqrt{64 + 36}$$

$$= \sqrt{100}$$

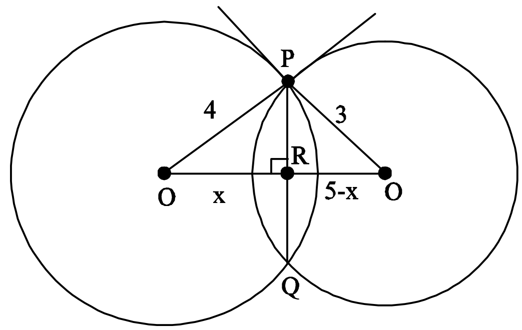
$$= 10 \text{ cm}$$

$\therefore$ Radius of the larger circle = 10 cm

---

8. Two circles with centers O and O' of radii 3 cm and 4 cm, respectively intersect at two point P and Q, such that OP and $O'P$ are tangents to the two circles. Find the length of the common chord PQ.

✍ **Solution:**



Given $OP = 4$ cm (radius of 1st circle)

$O'P = 3$ cm (radius of 2nd circle)

Clearly $OP \perp O'P$ (tangent and radius are $\perp$)

$$\therefore \quad OO' = \sqrt{4^2 + 3^2}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$= 5$$

Let R be a point of PQ such that

$OR = x$ and $O'R = 5 - x$

Also, $\Delta\, OPO' \sim \Delta\, OQO'$ and

$\Delta\, OPR \sim \Delta\, OQR$ (by similarity)

$$\therefore \quad \angle ORP = 90°$$

$\therefore$ In ΔORP, $PR^2 = 16 - x^2$

In $\Delta\, O'RP$, $PR^2 = 9 - (5 - x)^2$

$$\therefore \quad 16 - x^2 = 9 - (5 - x)^2$$

$$16 - x^2 = 9 - (25 - 10x + x^2)$$

$$16 - x^2 = 9 - 25 + 10x - x^2$$

$$16 - 9 + 25 = 10x$$

$$32 = 10x$$

$$x = \frac{32}{10}$$

$$x = \frac{16}{5}$$

$$\therefore \quad PR = \sqrt{16 - \frac{256}{25}}$$

$$= \sqrt{\frac{144}{25}} = \frac{12}{5}$$

$$= 2.4$$

$$\therefore \quad PQ = 2\,(PR)$$

$$= 2\,(2.4)$$

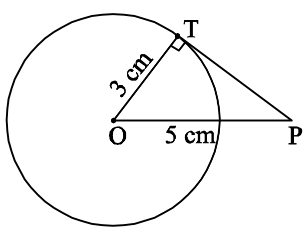
$$= 4.8 \text{ cm}$$

Example 4.24

Find the length of the tangent drawn from a point whose distance from the centre of a circle is 5 cm and radius of the circle is 3 cm.

✍ **Solution:**

Given $OP = 5$ cm, radius $r = 3$ cm



To find the length of tangent PT.

In right angled $\Delta\, OTP$,

$OP^2 = OT^2 + PT^2$ (by Pythagoras theorem)

$5^2 = 3^2 + PT^2$ gives $PT^2 = 25 - 9 = 16$

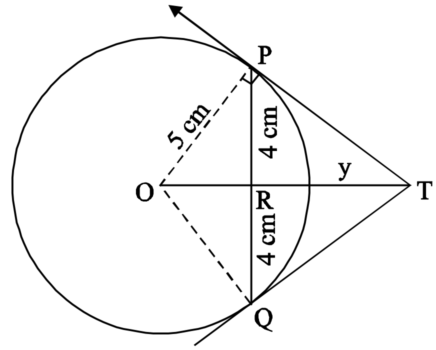
Length of the tangent $PT = 4$ cm

Example 4.25

PQ is a chord of length 8 cm to a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length of the tangent TP.

✍ **Solution:**

Let $TR = y$. Since, OT is perpendicular bisector of PQ.



$PR = QR = 4$ cm

In $\Delta\, ORP$, $OP^2 = OR^2 + PR^2$

$OR^2 = OP^2 - PR^2$

$OR^2 = 5^2 - 4^2 = 25 - 16 = 9 \Rightarrow OR = 3$ cm

$OT = OR + RT = 3 + y$ ...(1)

In $\Delta\, PRT$, $TP^2 = TR^2 + PR^2$ ...(2)

and $\Delta\, OPT$ we have, $OT^2 = TP^2 + OP^2$

$OT^2 = (TR^2 + PR^2) + OP^2$ (substitute for TP^2 from (2))

$(3 + y)^2 = y^2 + 4^2 + 5^2$ (substitute for OT from (1))

$9 + 6y + y^2 = y^2 + 16 + 25$

Therefore $y = TR = \frac{16}{3}$

$6y = 41 - 9$ we get $y = \frac{16}{3}$

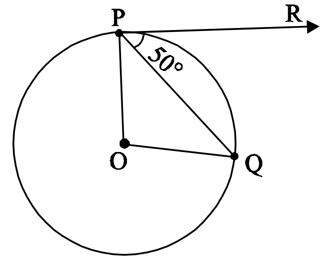
From (2), $TP^2 = TR^2 + PR^2$

$$TP^2 = \left(\frac{16}{3}\right)^2 + 4^2 = \frac{256}{9} + 16 = \frac{400}{9} \text{ so,}$$

$$TP = \frac{20}{3} \text{ cm}$$

Example 4.26

In figure O is the centre of a circle. PQ is a chord and the tangent PR and P makes an angle of 50° with PQ. Find ∠POQ



✍ **Solution:**

$\angle OPQ = 90° - 50° = 40°$ (angle between the radius and tangent is 90°)

$OP = OQ$ (Radii of a circle are equal)

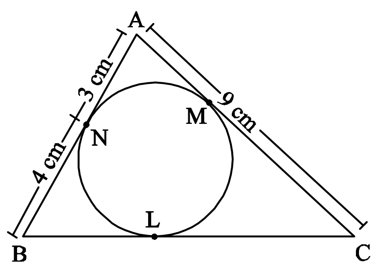
$\angle OPQ = \angle OQP = 40°$ ($\Delta\ OPQ$ is isosceles)

$\angle POQ = 180° - \angle OPQ - \angle OQP$

$\angle POQ = 180° - 40° - 40° = 100°$

Example 4.27

In Fig., ΔABC is circumscribing a circle. Find the length of BC.



✍ **Solution:**

$AN = M = 3$ cm (Tangents drawn from same external point are equal)

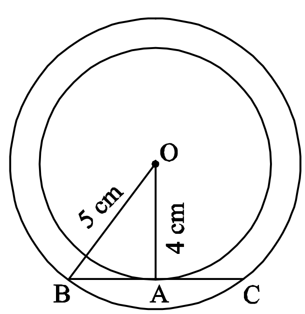
$BN = BL = 4$ cm

$CL = CM = AC - AM = 9 - 3 = 6$ cm

Gives $BC = BL + CL + 4 + 6 = 10$ cm

Example 4.28

If radii of two concentric circles are 4 cm and 5 cm then find the length of the chord of one circle which is a tangent to the other circle.



✍ **Solution:**

$OA = 4$ cm, $OB = 5$ cm ; also $OA \perp BC$

$OB^2 = OA^2 + AB^2$

$5^2 = 4^2 + AB^2$ gives $AB^2 = 9$

Therefore $AB = 3$ cm

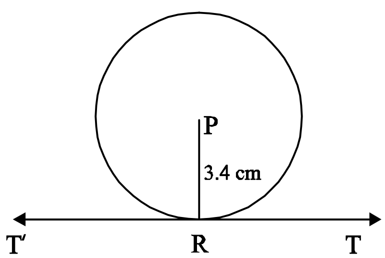
$BC = 2AB$ hence $BC = 2 \times 3 = 6$ cm

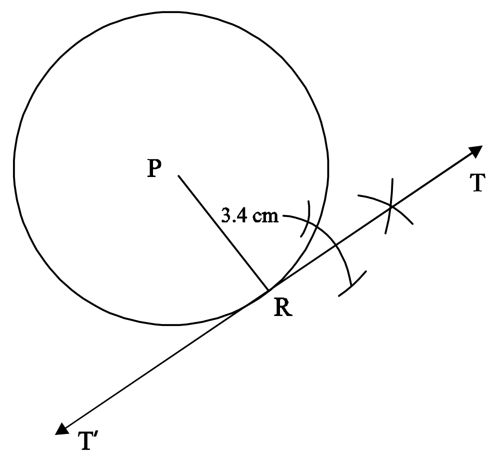
Type II: (Construction of a tangent)

I. Using centre one tangent
Q.No. 12, Example 4.29

12. Draw a tangent at any point *R* on the circle of radius 3.4 cm and centre at *P*?

✍ **Solution:**





Construction

Step 1: Draw a circle with centre at *P* of radius 3.4 cm.

Step 2: Take a point *R* on the circle and Join *PR*.

Step 3: Draw perpendicular line *TT′* to *PR* which passes through *R*.

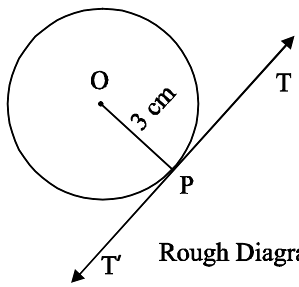
Step 4: *TT′* is the required tangent.

Example 4.29

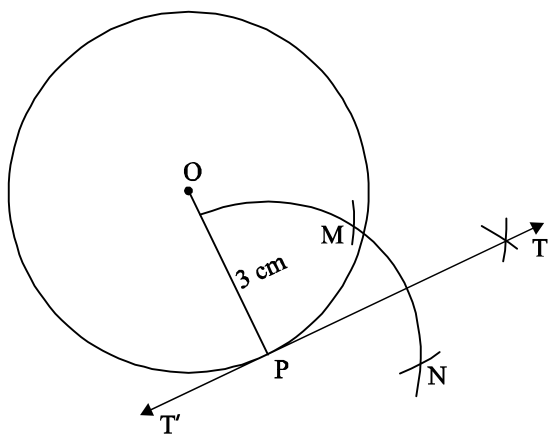
Draw a circle of radius 3 cm. Take a point P on this circle and draw a tangent at P.

✍ **Solution:**

Given, radius $r = 3$ cm



Rough Diagram



Construction

Step 1: Draw a circle with centre at O of radius 3 cm.

Step 2: Take a point P on the circle. Join OP.

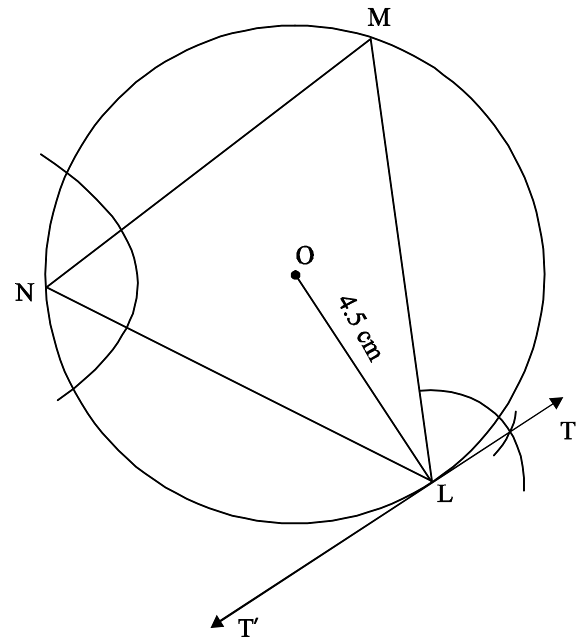
Step 3: Draw perpendicular line TT' to OP which passes through P.

Step 4: TT' is the required tangent.

II. Using alternate segment theorem Q.No. 13, Example 4.30

13. Draw a circle of radius 4.5 cm. Take a point on the circle. Draw the tangent at that point using the alternate segment theorem.

✍ **Solution:**



Construction

Step 1: With O as the centre, draw a circle of radius 4.5 cm.

Step 2: Take a point L on the circle. Through L draw any chord LM.

Step 3: Take a point M distinct from L and N on the circle, so that L, M and N are in anti-clockwise direction. Join LN and NM.

Step 4: Through L draw a tangent TT' such that $\angle TLM = \angle MNL$

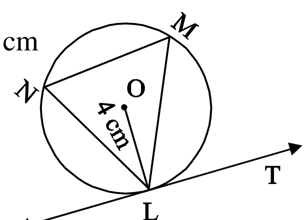
Step 5: TT' is the required tangent.

Example 4.30

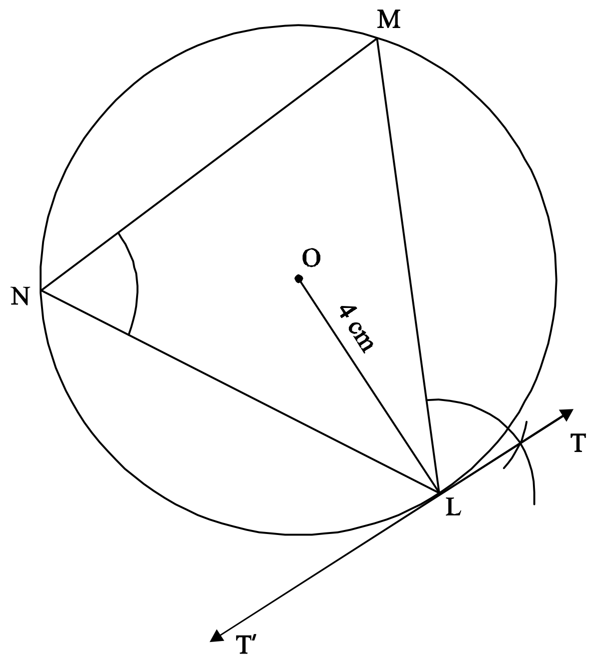
Draw a circle of radius 4 cm. At a point L on it draw a tangent to the circle using the alternate segment.

✍ **Solution:**

Given, radius = 4 cm



Rough Diagram



Construction

Step 1: With O as the centre, draw a circle of radius 4 cm.

Step 2: Take a point L on the circle. Through L draw any chord LM.

Step 3: Take a point M distinct from L and N on the circle, so that L, M and N are in anticlockwise direction. Join LN and NM.

Step 4: Through L draw a tangement TT' such that $\angle TLM = \angle MNL$.

Step 5: TT' is the required tangent.

III. Using centre two tangent
Q.No. 14, 15, 16, 17, Example 4.31.

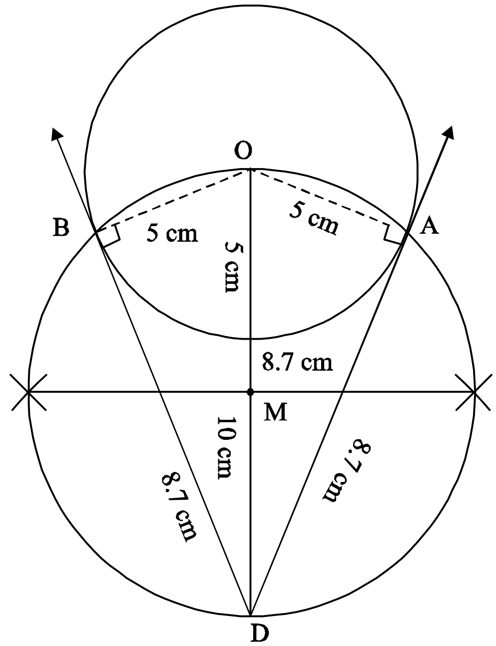
14. Draw the two tangents from a point which is 10 cm away from the centre of a circle of radius 5 cm. Also, measure the lengths of the tangents.

✍ **Solution:**

Construction

Step 1: With centre at O, draw a circle of radius 5 cm.

Step 2: Draw a line $OP = 10$ cm



Step 3: Draw a perpendicular bisector of OP, which cuts OP at M.

Step 4: With M as centre and MO as radius, draw a circle which cuts previous circle at A and B.

Step 5: Join AP and BP. AP and BP are the required tangents. Thus length of the tangents are $PA = PB = 8.7$ cm

Verification: In the right angle triangle $\Delta\, OAP$,

$$PA^2 = \sqrt{OP^2 - OA^2}$$

$$= \sqrt{100 - 25} = \sqrt{75} = 8.7 \text{ cm}$$

15. Take a point which is 11 cm away from the centre of a circle of radius 4 cm and draw the two tangents to the circle from that point.

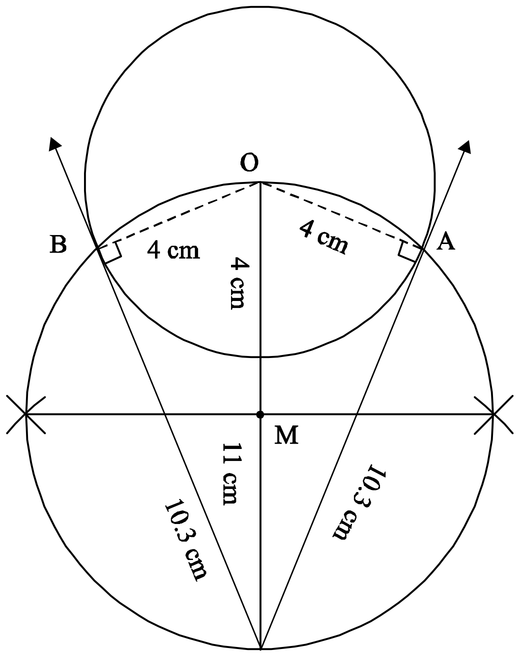
✍ **Solution:**

Construction

Step 1: With centre at O, draw a circle of radius 4 cm.

Step 2: Draw a line $OP = 11$ cm

Step 3: Draw a perpendicular bisector of OP, which cuts OP at M.



Step 4: With M as centre and MO as radius, draw a circle which cuts previous circle at A and B.

Step 5: Joint AP and BP. They are the required tangents $AP = BP = 10.3$ cm

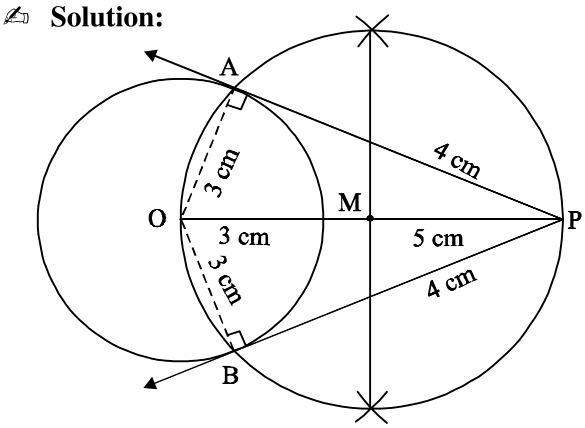
Verification: In the right angle triangle $\Delta\, OAP$,

$$AP = \sqrt{OP^2 - OA^2}$$

$$= \sqrt{121 - 16} = \sqrt{105} = 10.3 \text{ cm}$$

16. Draw the two tangents from a point which is 5 cm away from the centre of a circle of diameter 6 cm. Also, measure the lengths of the tangents.

✍ **Solution:**



Construction

Step 1: With centre at O, draw a circle of radius 3 cm. with centre at O.

Step 2: Draw a line $OP = 5$ cm

Step 3: Draw a perpendicular bisector of OP, which cuts OP at M.

Step 4: With M as centre and OM as radius, draw a circle which cuts previous circle at A and B.

Step 5: Join AP and BP. They are the required tangents $AP = BP = 4$ cm

Verification:

$$AP = \sqrt{OP^2 - OA^2}$$

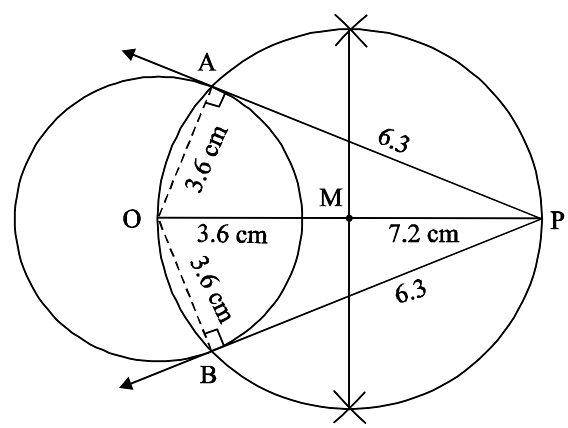
$$= \sqrt{5^2 - 3^2}$$

$$= \sqrt{25 - 9}$$

$$= \sqrt{16} = 4 \text{ cm}$$

17. Draw a tangent to the circle from the point P having radius 3.6 cm, and centre at O. Point P is at a distance 7.2 cm from the centre.

✍ **Solution:**



Construction

Step 1: Draw a circle of radius 3.6 cm. with centre at O.

Step 2: Draw a line $OP = 7.2$ cm

Step 3: Draw a perpendicular bisector of OP, which cuts it M.

Step 4: With M as centre and OM as radius, draw a circle which cuts previous circle at A and B.

Step 5: Join AP and BP. They are the required tangents $AP = BP = 6.3$ cm

Verification:

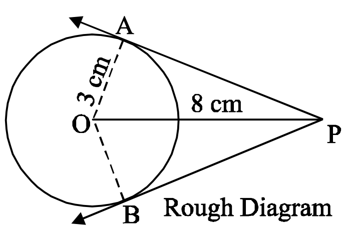
$$AP = \sqrt{OP^2 - OA^2}$$
$$= \sqrt{(7.2)^2 - (3.6)^2}$$
$$= \sqrt{51.84 - 12.96}$$
$$= \sqrt{38.88} = 6.3 \text{ (approx)}$$

Example 4.31

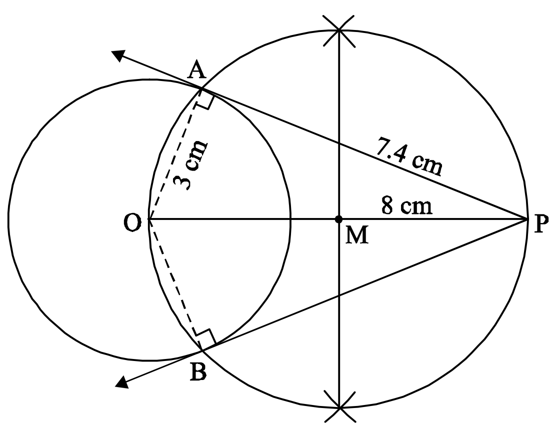
Draw a circle of diameter 6 cm from a point P, which is 8 cm away from its centre. Draw the two tangents PA and PB to the circle and measure their lengths.

✍ **Solution:**

Given, diameter $(d) = 6$ cm, we find radius $(r) = \frac{6}{2} = 3$ cm



Rough Diagram



Construction

Step 1: With centre at O, draw a circle of radius 3 cm.

Step 2: Draw a line OP of length 8 cm.

Step 3: Draw a perpendicular bisector of OP, which cuts OP at M.

Step 4: With M as centre and MO as radius, draw a circle which cuts previous circle at A and B.

Step 5: Join AP and BP. AP and BP are the required tangents. Thus length of the tangents are $PA = PB = 7.4$ cm

Verification: In the right angle triangle OAP,

$$PA^2 = OP^2 - OA^2 = 64 - 9 = 55$$

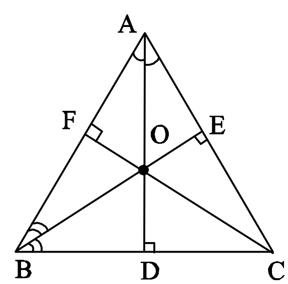
$PA = \sqrt{55} = 7.4$ cm (approximately)

Type III: Concurrency theorems

Q.No. 9. Example 4.32, 10, 11, Example 4.33, 4.34, 4.35

9. Show that the angle bisectors of a triangle are concurrent.

✍ **Solution:**



Consider a ΔABC and let the angular bisectors of A and B meet at 'O'.

From O, draw perpendicular OD, OE, OF to BC, CA, AB respectively.

Now $\Delta BOD = BOF$

$(\because \angle ODB = \angle OFB = 90° \; \angle OBD = \angle OBF)$

$\therefore \quad OD = OF$

IIIy in ΔOAE and ΔOAF, we can prove

$$OE = OF$$

$\therefore \quad OD = OE = OF$

Now, join OC,

Consider $\Delta\, OCD, \Delta\, OCE$

Here (i) $\angle ODC = \angle OEC = 90°$ and OC is common

(ii) $OD = OE$

$\therefore \quad \Delta\, OCD = \Delta\, OCE$

$\therefore \quad \angle OCD = \angle OCE$

CO is angle bisector of $\angle C$

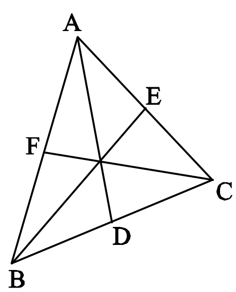
$\therefore$ Angle bisectors of a triangle are concurrent.

Example 4.32

Show that in a triangle, the medians are concurrent.

✍ **Solution:**

Medians are line segments joining each vertex to the midpoint of the corresponding opposite sides.



Thus medians are the cevians where D, E, F are midpoints of BC, CA and AB respectively.

Since D is a mid point of

$$BC, BD = DC \text{ so } \frac{BD}{DC} = 1 \quad ...(1)$$

Since, E is a midpoint of

$$CA, CE = EA \text{ so } \frac{CE}{EA} = 1 \quad ...(2)$$

Since, F is a midpoint of AB,

$$AB, AF = FB \text{ so } \frac{AF}{FB} = 1 \quad ...(3)$$

Thus, multiplying (1), (2) and (3) we get,

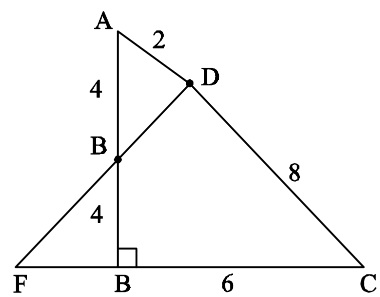
$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1 \times 1 \times 1 = 1$$

And so, Ceva's theorem is satisfied.

Hence the Medians are concurrent.

10. In ΔABC, with $B = 90°, BC = 6$ cm and $AB = 8$ cm, D is a point on AC such that $AD = 2$ cm and E is the midpoint of AB. Join D to E and extend it to meet at F. Find BF.

✍ **Solution:**



Given In $\Delta\, ABC, AB = 8$ cm, $BC = 6$ cm

$\therefore \quad AC = \sqrt{64 + 36} = \sqrt{100} = 10$

Also $AD = 2 \Rightarrow CD = 8$ cm

E is the mid point of AB

$\Rightarrow \quad AE = EB = 4$ cm

By Menelaus Theorem,

$$\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CD}{DA} = 1$$

$$\Rightarrow \quad \frac{4}{4} \times \frac{BF}{BF + 6} \times \frac{8}{2} = 1$$

$$\Rightarrow \quad 4BF = BF + 6$$

$$\Rightarrow \quad 3BF = 6$$

$$\therefore \quad BF = 2 \text{ cm}$$

CHAPTER 5

CO-ORDINATE GEOMETRY

Exercise 5.1

KEY POINTS

Recall: (IX-Std formulae)

1. Distance between $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$|AB| = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. Midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

3. Section formula:

(i) If a point $P(x, y)$ divide the line segment $A(x_1, y_1)$ and $B(x_2, y_2)$ **in-ternally** in the ratio $m : n$ then

$$P = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$$

(ii) Externally $P = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}\right)$

4. Centroid of a triangle whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Area of a triangle

1. **Area of a triangle whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$**

$$A = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

sq.units.

Another form

$$A = \frac{1}{2}\left\{\begin{matrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{matrix}\right\}$$

$$= \frac{1}{2}\left[(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3)\right]$$

2. **Area of a Quadrilateral whose vertices are $A(x_1, y_1), B(x_2, y_2)$ $C(x_3, y_3)$ and $D(x_4, y_4)$.**

$$\text{Area} = \frac{1}{2}\left[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\right]$$

sq.units.

Another form

$$A = \frac{1}{2}\left\{\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{matrix}\right\}$$

$$= \frac{1}{2}\left\{(x_1 y_2 + x_2 y_3 + x_3 y_4 + x_4 y_1) - (x_2 y_1 + x_3 y_2 + x_4 y_3 + x_1 y_4)\right\} \text{ sq.units.}$$

Note

(i) If area of the triangle is **zero**, then given points are collinear (lie on the same line).

If $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear points, then

- $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

(Or)

- $x_1 y_2 + x_2 y_3 + x_3 y_1 = x_1 y_3 + x_2 y_1 + x_3 y_2$

(ii) The area of the triangle and quadrilateral is never negative. That is, we always take the area as positive. (Absolute value)

(iii) Plot the points in the Rough Graph, and take anticlockwise points as order. (Here always area will come positive)

Type I: Area of triangle based sums.

Q.No: 1. (i) (ii), Example 5.1, 5.2, 2. (i) (ii), 3. (i) (ii), Example 5.3, 5.4, 4. (i) (ii), 7.

1. Find the area of the triangle formed by the points

(i) $(1, -1), (-4, 6)$ and $(-3, -5)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} 1 & -4 & -3 & 1 \\ -1 & 6 & -5 & -1 \end{bmatrix}$$

$$= \frac{1}{2}[\,(6 + 20 + 3) - (4 - 18 - 5)\,]$$

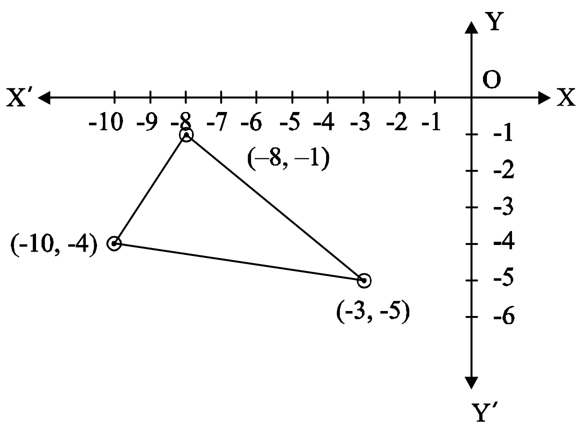
$$= \frac{1}{2}[(29) - (-19)]$$

$$= \frac{1}{2}[29 + 19]$$

$$= \frac{1}{2} \times 48$$

$$= 24 \text{ sq.units}$$

(ii) $(-10, -4), (-8, -1)$ and $(-3, -5)$



Here $A(-8, -1)$, $B(-10, -4)$, $C(-3, -5)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -8 & -10 & -3 & -8 \\ -1 & -4 & -5 & -1 \end{bmatrix}$$

$$= \frac{1}{2}[\,(32 + 50 + 3) - (10 + 12 + 40)\,]$$

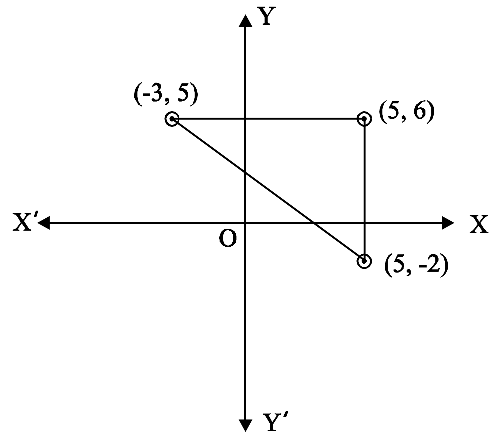
$$= \frac{1}{2}[(85) - (62)]$$

$$= \frac{1}{2}[23]$$

$$= 11.5 \text{ sq.units}$$

Example 5.1

Find the area of the triangle whose vertices are $(-3, 5), (5, 6)$ and $(5, -2)$.



Here

Let $A(5, 6)$, $B(-3, 5)$ and $C(5, -2)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} 5 & -3 & 5 & 5 \\ 6 & 5 & -2 & 6 \end{bmatrix}$$

$$= \frac{1}{2}[\,(25 + 6 + 30) - (-18 + 25 - 10)\,]$$

$$= \frac{1}{2}[(61) - (-3)]$$

$$= \frac{1}{2}[61 + 3]$$

$$= \frac{1}{2} \times 64$$

$$= 32 \text{ sq.units}$$

Example 5.2

Show that the points $P(-1.5, 3)$, $Q(6, -2)$ $R(-3, 4)$ are collinear.

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -1.5 & 6 & -3 & -1.5 \\ 3 & -2 & 4 & 3 \end{bmatrix}$$

$$= \frac{1}{2}[(3 + 24 - 9) - (18 + 6 - 6)]$$

$$= \frac{1}{2}[(18) - (18)]$$

$$\text{Area} = 0$$

Given points are collinear.

2. Determine whether the sets of points are collinear?

(i) $\left(-\frac{1}{2}, 3\right)$**, (– 5, 6) and (– 8, 8)**

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -1/2 & -5 & -8 & -1/2 \\ 3 & 6 & 8 & 3 \end{bmatrix}$$

$$\text{Area} = \frac{1}{2}[(-3-40-24)-(-15-48-4)]$$

$$= \frac{1}{2}[(-67)-(-67)]$$

$$= \frac{1}{2}[-67+67]$$

$$\text{Area} = 0$$

$\therefore$ Given points are collinear.

(ii) $(a, b+c), (b, c+a)$ **and** $(c, a+b)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} a & b & c & a \\ b+c & c+a & a+b & b+c \end{bmatrix}$$

$$= \frac{1}{2}\left[(ac+a^2+ab+b^2+bc+c^2) - (b^2+bc+c^2+ac+a^2+ab)\right]$$

$$= \frac{1}{2}\left[(a^2+b^2+c^2+ab+bc+ca) - (a^2+b^2+c^2+ab+bc+ca)\right]$$

$$= \frac{1}{2}(0)$$

$$\text{Area} = 0$$

$\therefore$ Given points are collinear.

---

3. Vertices of given triangles are taken in order and their areas are provided aside. In each case, find the value of 'p'.

S.No.	Vertices	Area (sq.Units)
(i)	(0,0),(p,8),(6,2)	20
(ii)	(p,p),(5,6),(5-2)	32

(i) (0,0),(p,8),(6,2) Area = 20 sq.units.

$$\text{Area} = \frac{1}{2}\begin{bmatrix} 0 & p & 6 & 0 \\ 0 & 8 & 2 & 0 \end{bmatrix}$$

$$20 = \frac{1}{2}[(0+2p+0)-(0+48+0)]$$

$$20 \times 2 = 2p - 48$$

$$40 = 2p - 48$$

$$40 + 48 = 2p$$

$$88 = 2p$$

$$\frac{88}{2} = p$$

$$\boxed{44 = p}$$

(ii) (p,p), (5,6), (5, – 2) Area = 32 sq.units.

$$\text{Area} = \frac{1}{2}\begin{bmatrix} p & 5 & 5 & p \\ p & 6 & -2 & p \end{bmatrix}$$

$$32 = \frac{1}{2}[(6p-10+5p)-(5p+30-2p)]$$

$$32 \times 2 = 6p - 10 + 5p - 5p - 30 + 2p$$

$$64 = 8p - 40$$

$$64 + 40 = 8p$$

$$104 = 8p$$

$$\frac{104}{8} = p$$

$$\boxed{13 = p}$$

Example 5.3

If the area of the triangle formed by the vertices A (– 1, 2), B (k, – 2) and C (7, 4) (taken in order) is 22 sq.units. Find the value of K.

$A(-1, 2), B(k, -2), C(7, 4)$

Area $= 22$ sq.units.

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -1 & K & 7 & -1 \\ 2 & -2 & 4 & 2 \end{bmatrix}$$

$$22 = \frac{1}{2}[(2+4k+14)-(2k-14-4)]$$

$$22 \times 2 = 2 + 4k + 14 - 2k + 14 + 4$$

$$44 = 2k + 34$$

$$44 - 34 = 2k$$

$$10 = 2k$$

$$\frac{10}{2} = k$$

$$\boxed{5 = k}$$

Example 5.4

If the points $P(-1,-4), Q(b,c)$ and $R(5,-1)$ are collinear and if $2b+c=4$, then find the values of b and c.

$P(-1,-4), Q(b,c), R(5,-1)$

Given points are collinear

$$\begin{bmatrix} -1 & b & 5 & -1 \\ -4 & c & -1 & -4 \end{bmatrix}$$

Product of downward values = product of upward values.

$$-c-b-20 = -4b+5c+1$$

$$-c-b+4b-5c = 1+20$$

$$3b-6c = 21 \quad ...(1)$$

Given

$$2b+c = 4 \quad ...(2)$$

Solve (1) and (2)

$$(1) \Rightarrow 3b - bc = 21$$

$$(2) \times 5 \Rightarrow 12b + 6c = 24$$

$$15b = 45$$

$$b = \frac{45}{15}$$

$$\boxed{b = 3}$$

$$(2) \Rightarrow 2(3) + c = 4$$

$$6 + c = 4$$

$$c = 4 - 6$$

$$\boxed{c = -2}$$

7. If the points $A(-3,9), B(a,b)$ and $C(4,-5)$ are collinear and if $a+b=1$) then find a and b.

Given points are collinear.

$$\begin{bmatrix} -3 & a & 4 & -3 \\ 9 & b & -5 & 9 \end{bmatrix}$$

$$-3b-5a+36 = 9a+4b+15$$

$$36-15 = 9a+4b+3b+5a$$

$$21 = 14a+7b$$

divide by 7

$$2a+b = 3 \quad ...(1)$$

Given $\quad a+b = 1 \quad ...(2)$

Solve (1) & (2)

$$2a+b = 3$$

$$(-) \quad (-) \quad (-)$$

$$a+b = 1$$

$$a = 2$$

$$(2) \Rightarrow 2+b = 1$$

$$b = 1-2$$

$$\boxed{b = -1}$$

4. In each of the following find the value of 'a' for which the given points are collinear. (i) (2,3), (4,a) and (6, – 3)

Given points are collinear.

$$\begin{bmatrix} 2 & 4 & 6 & 2 \\ 3 & a & -3 & 3 \end{bmatrix}$$

$$2a-12+18 = 12+6a-6$$

$$2a+6 = 6a+6$$

$$2a-6a = 6-6$$

$$-4a = 0$$

$$\boxed{a = 0}$$

(ii) $(a, 2-2a), (-a+1, 2a)$ and $(-4-a, 6-2a)$.

Given points are collinear.

$$\begin{bmatrix} a & -a+1 & -4-a & a \\ 2-2a & 2a & 6-2a & 2-2a \end{bmatrix}$$

$a(2a) + (6-2a)(-a+1) + (2-2a)(-4-a)$

$= (2-2a)(-a+1) + 2a(-4-a) + a(6-2a)$

$2a^2 - 6a + 6 + 2a^2 - 2a - 8 - 2a + 8a + 2a^2$

$= -2a + 2 + 2a^2 - 2a - 8a - 2a^2 + 6a - 2a^2$

$6a^2 - 2a - 2 = -2a^2 - 6a + 2$

$6a^2 - 2a - 2 + 2a^2 + 6a - 2 = 0$

$8a^2 + 4a - 4 = 0$

$2a^2 + a - 1 = 0$

$(a+1)(2a-1) = 0$

-2

$2 \quad -1$

$\frac{2}{2}, \frac{-1}{2}$

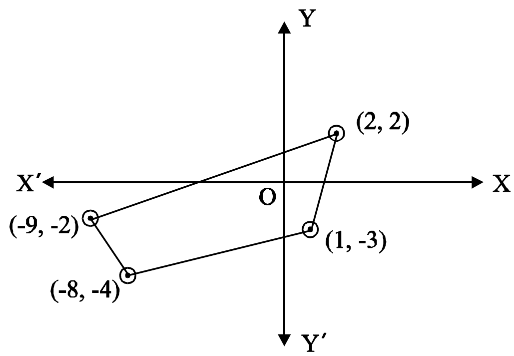
$1, \frac{-1}{2}$

$a + 1 = 0$	$2a - 1 = 0$
$\boldsymbol{a = -1}$	$2a = 1$
	$\boldsymbol{a = \frac{1}{2}}$

Type: II Area of Quadrilateral based sums.
Q.No: 5. (i) (ii), Example 5.6, 6

5. Find the area of the Quadrilateral whose vertices are at

(i) (– 9, – 2), (– 8, – 4), (2, 2) and (1, – 3)



Here

Let $A(2, 2), B(-9, -2), C(-8, -4)$ and $D(1, -3)$

$$\text{Area} = \frac{1}{2}\begin{Bmatrix} 2 & -9 & -8 & 1 & 2 \\ 2 & -2 & -4 & -3 & 2 \end{Bmatrix}$$

$= \frac{1}{2}[(-4 + 36 + 24 + 2) - (-18 + 16 - 4 - 6)]$

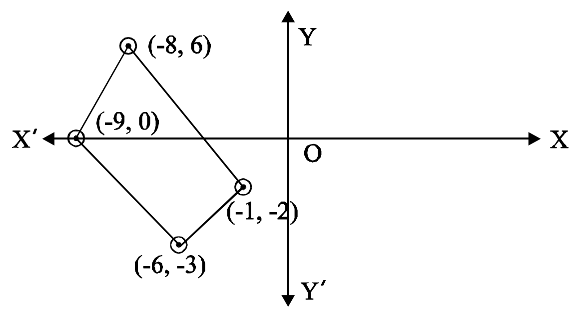
$= \frac{1}{2}[(58) - (-12)]$

$= \frac{1}{2}[58 + 12]$

$= \frac{1}{2} \times 70$

$= 35$ sq.units

(ii) (– 9, 0), (– 8, 6), (– 1, – 2) and (– 6, – 3)



Here Let

$A(-8, 6), B(-9, 0), C(-6, -3)$ and $D(-1, -2)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -8 & -9 & -6 & -1 & -8 \\ 6 & 0 & -3 & -2 & 6 \end{bmatrix}$$

$= \frac{1}{2}[(0 + 27 + 12 - 6) - (-54 + 0 + 3 + 16)]$

$= \frac{1}{2}[(33) - (-35)]$

$= \frac{1}{2}[33 + 35]$

$= \frac{1}{2} \times 68$

$= 34$ sq.units

6. Find the value of k if the area of a Quadrilateral is 28 sq.units whose vertices are $(-4, -2), (-3, k), (3, -2)$ and $(2, 3)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -4 & -3 & 3 & 2 & -4 \\ -2 & k & -2 & 3 & -2 \end{bmatrix}$$

$28 = \frac{1}{2}[(-4k + 6 + 9 - 4) - (6 + 3k - 4 - 12)]$

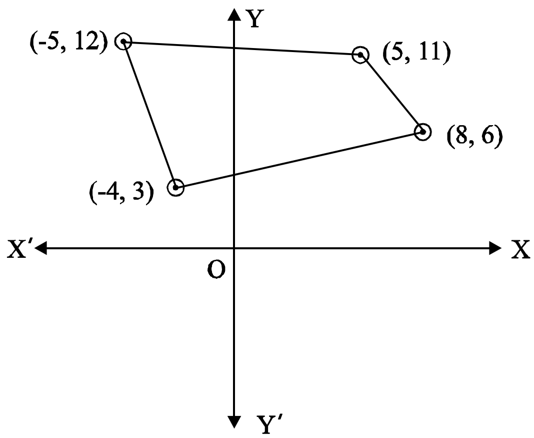
$28 \times 2 = (-4k + 11) - (-10 + 3k)$

$56 = -4k + 11 + 10 - 3k$

$$56 = -7k + 21$$
$$56 - 21 = -7k$$
$$35 = -7k$$
$$\frac{35}{-7} = k$$
$$\boxed{-5 = k}$$

Example 5.6

Find the area of the Quadrilateral formed by the points (8,6), (5,11), (− 5, 12) and (− 4, 3)



Let $A(5, 11), B(-5, 12), C(-4, 3)$ and $D(8, 6)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} 5 & -5 & -4 & 8 & 5 \\ 11 & 12 & 3 & 6 & 11 \end{bmatrix}$$
$$= \frac{1}{2}[(60 - 15 - 24 + 88) - (-55 - 48 + 24 + 30)]$$
$$= \frac{1}{2}[(109) - (-49)]$$
$$= \frac{1}{2}(109 + 49)$$
$$= \frac{1}{2} \times 158$$
$$= 79 \text{ sq.units}$$

Type: III Word problems based on area concept.

Q.No: Example 5.5, 5.7 9. 10, 11, 8

Example 5.5

The floor of a hall is covered with identical tiles which are in the shapes of triangles. One such triangle has the vertices at (− 3, 2), (− 1, − 1) and (1,2). If the floor of the hall is completely covered by 110 tiles. Find the area of the floor.

Vertices of one triangular tile are at $(-3, 2), (-1, -1)$ and $(1,2)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -3 & -1 & 1 & -3 \\ 2 & -1 & 2 & 2 \end{bmatrix}$$
$$= \frac{1}{2}[(3 - 2 + 2) - (-2 - 1 - 6)]$$
$$= \frac{1}{2}[(3) - (-9)]$$
$$= \frac{1}{2}(3 + 9)$$
$$= \frac{1}{2} \times 12$$
$$= 6 \text{ sq.units}$$

No.of tiles covered in the floor $= 110$

Total Area of the floor $= 110 \times 6$

$= 660$ sq.units

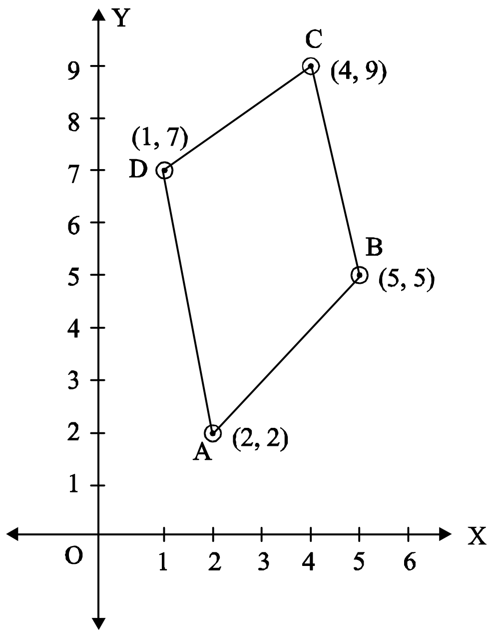
Example 5.7

The given diagram shows a plan for constructing a new parking lot at a campus. It is estimated that such construction would cost Rs.1300 per square feet what will be the total cost for making the parking lot.

The parking lot is a quadrilateral whose vertices at $A(2, 2), B(5, 5), C(4, 9)$ and $D(1, 7)$

$\therefore$ Area of parking lot

$$A = \frac{1}{2}\begin{bmatrix} 2 & 5 & 4 & 1 & 2 \\ 2 & 5 & 9 & 7 & 2 \end{bmatrix}$$



$$= \frac{1}{2}[(10+45+28+2)-(10+20+9+14)]$$

$$= \frac{1}{2}[85-53]$$

$$= \frac{1}{2}(32)$$

$= 16$ sq units

Construction rate per sq.feet $= Rs.1300$

Total cost $= 1300 \times 16$

$= \text{Rs.}20{,}800$

10. A triangular shaped glass with vertices at $A(-5,-4)$, $B(1,6)$ and $C(7,-4)$ has to be painted. If one bucket of paint covers 6 square feets, how many buckets of paint will be required to paint the whole glass, if only one coat of paint is applied.

Vertices of the triangular glass

$A(-5,-4)$, $B(1,6)$, $C(7,-4)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -5 & 1 & 7 & -5 \\ -4 & 6 & -4 & -4 \end{bmatrix}$$

$$= \frac{1}{2}[(-30-4-28)-(-4+42+20)]$$

$$= \frac{1}{2}[(-62)-(58)]$$

$$= \frac{1}{2}[-62-58]$$

$$= \frac{1}{2}(-120)$$

$= 60$ sq units

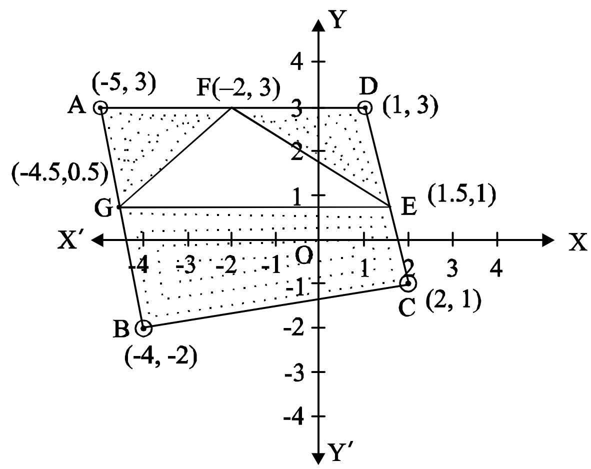
[Area +ve only]

6 square feets covers of paint $= 1$ bucket

60 square feet covers of paint $= \frac{60}{6}$

$= 10$ buckets

10. In the figure find the area of
(i) triangle AGF (ii) triangle FED
(iii) Quadrilateral BCEG.



(i) Area of triangle AGF

$A(-5,3)$ $G(-4.5,0.5)$ $F(-2,3)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -5 & -4.5 & -2 & -5 \\ 3 & 0.5 & 3 & 3 \end{bmatrix}$$

$$= \frac{1}{2}[(-2.5-13.5-6)-(-13.5-1-15)]$$

$$= \frac{1}{2}[(-22)-(-29.5)]$$

$$= \frac{1}{2}[-22+29.5]$$

$$= \frac{2}{2} \times 7.5$$

$= 3.75$ sq.units

(ii) Area of triangle FED

$F(-2,3)$ $E(1.5,1)$ $D(1,3)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -2 & 1.5 & 1 & -2 \\ 3 & 1 & 3 & 3 \end{bmatrix}$$

$$= \frac{1}{2}[(-2+4.5+3)-(4.5+1-6)]$$

$$= \frac{1}{2}[(5.5)-(-0.5)]$$

$$= \frac{1}{2}[5.5+0.5]$$

$$= \frac{1}{2}\times 6$$

$$= 3 \text{ sq.units}$$

(iii) Area of quadrilateral BCEG

$B(-4,-2)\ C(2,-1)\ E(1.5, 1),.\ G(-4.5, 0.5)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -4 & 2 & 1.5 & -4.5 & -4 \\ -2 & -1 & 1 & 0.5 & -2 \end{bmatrix}$$

$$= \frac{1}{2}[(4+2+0.75+9)-(-4-1.5-4.5-2)]$$

$$= \frac{1}{2}[(15.75)-(-12)]$$

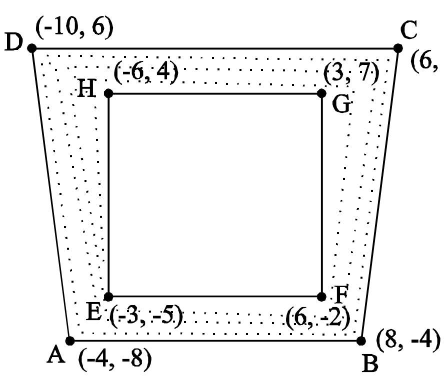
$$= \frac{1}{2}[15.75+12]$$

$$= \frac{1}{2}[27.75]$$

$$= 13.875$$

$$= 13.88 \text{ sq.units}$$

9. In the figure, the Quadrilateral swimmingpool shown is surrounded by concrete patio. Find the area of the patio.



- Area of quadrilateral ABCD

$A(-4,-8), B(8,-4),\ \ C(6, 10), D(-10, 6)$

$$\text{Area} = \begin{bmatrix} -4 & 8 & 6 & -10 & -4 \\ -8 & -4 & 10 & 6 & -8 \end{bmatrix}$$

$$= \frac{1}{2}[(16+80+36+80)-(-64-24-100-24)]$$

$$= \frac{1}{2}[(212)-(-212)]$$

$$= \frac{1}{2}(212+212)$$

$$= \frac{424}{2}$$

$$= 212 \text{ sq.units}$$

- Area of quadrilateral EFGH

$E(-3,-5) F(6,-2)\ G(3, 7)\ H(-6, 4)$

$$\text{Area} = \frac{1}{2}\begin{bmatrix} -3 & 6 & 3 & -6 & -3 \\ -5 & -2 & 7 & 4 & -5 \end{bmatrix}$$

$$= \frac{1}{2}[(6+42+12+30)-(-30-6-42-12)]$$

$$= \frac{1}{2}[(90)-(-90)]$$

$$= \frac{1}{2}(90+90)$$

$$= \frac{1}{2}\times 180$$

$$= 90 \text{ sq.units}$$

$\therefore$ Area of the patio = Area of quadrilateral ABCD - Area of Quadrilateral EFGH.

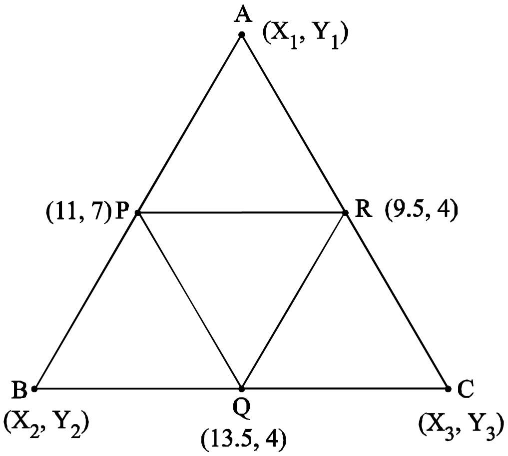
$$= 212 - 90$$

$$= 122 \text{ sq.units}$$

8. Let $P(11, 7), Q(13.5, 4)$ and $R(9.5, 4)$ be the midpoints of the sides AB, BC and AC respectively of ΔABC. Find the co-ordinates of the vertices A, B and C. Hence find the area of ΔABC and compare this with area of ΔPQR

- $P =$ mid point of AB

$$(11, 7) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$



- $Q =$ mid point of BC

$$(13.5, 4) = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

- $R =$ mid point of AC

$$(9.5, 4) = \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right)$$

Comparing 'x' values

$\frac{x_1 + x_2}{2} = 11$

$x_1 + x_2 = 22$ ...(1)

$\frac{x_2 + x_3}{2} = 13.5$

$x_2 + x_3 = 27$ ...(2)

$\frac{x_1 + x_3}{2} = 9.5$

$x_1 + x_3 = 19$ ...(3)

Add (1)(2) & (3)

$2(x_1 + x_2 + x_3) = 22 + 27 + 19$

$x_1 + x_2 + x_3 = \frac{68}{2}$

$x_1 + x_2 + x_3 = 34$ ...(4)

Put (1) in (4)

$22 + x_3 = 34$

$x_3 = 34 - 22$

$\mathbf{x_3 = 12}$

Put (2) in (4)

$x_1 + 27 = 34$

$x_1 = 34 - 27$

$\mathbf{x_1 = 7}$

Put (3) in (4)

$x_2 + 19 = 34$

$x_2 = 34 - 19$

$\mathbf{x_2 = 15}$

Comparing 'y values

$\frac{y_1 + y_2}{2} = 7$

$y_1 + y_2 = 14$ ...(1)

$\frac{y_2 + y_3}{2} = 4$

$y_2 + y_3 = 8$ ...(2)

$\frac{y_1 + y_3}{2} = 4$

$y_1 + y_3 = 8$ ...(3)

Add (1), (2)& (3)

$2(y_1 + y_2 + y_3) = 14 + 8 + 8$

$y_1 + y_2 + y_3 = \frac{30}{2}$

$y_1 + y_2 + y_3 = 15$ ...(4)

Put (1) in (4)

$14 + y_3 = 15$

$y_3 = 15 - 14$

$\mathbf{y_3 = 1}$

Put (2) in (4)

$y_1 + 8 = 15$

$y_1 = 15 - 8$

$\mathbf{y_1 = 7}$

Put (3) in (4)

$y_2 + 8 = 15$

$y_2 = 15 - 8$

$\mathbf{y_2 = 7}$

$\therefore A\,(7, 7)\; B\,(15, 7),\; C\,(12, 1)$

- Area of ΔABC

$$\text{Area} = \frac{1}{2}\begin{bmatrix} 7 & 15 & 12 & 7 \\ 7 & 7 & 1 & 7 \end{bmatrix}$$

$$= \frac{1}{2}[(49 + 15 + 84) - (105 + 84 + 7)]$$

$$= \frac{1}{2}[148 - 196]$$

$$= \frac{1}{2}(-48)$$

$= 24$ sq.units [Area not negative]

- Area of ΔPQR

$$\text{Area} = \frac{1}{2}\begin{bmatrix} 11 & 13.5 & 9.5 & 11 \\ 7 & 4 & 4 & 7 \end{bmatrix}$$

$$= \frac{1}{2}[(44 + 54 + 66.5) - (94.5 + 38 + 44)]$$

$$= \frac{1}{2}[164.5 - 176.5]$$

$$= \frac{1}{2}(-12)$$

$= 6$ sq.units [Area not negative]

Hence, we get

$\therefore$ Area of $\Delta ABC = 4$ (Area of ΔPQR)

Exercise 5.2

KEY POINTS

1. Inclination of a line

A straight line which makes an angle with x-axis measured in the counter - clockwise direction to the part of the line above the x-axis is called inclination of a line or the angle of inclination. It is denoted by θ

- The inclination of X-axis and every line parallel to X-axis is $0°$
- The inclination of Y-axis and every line parallel to Y-axis is $90°$

2. Slope of a line

The tangent value of angle of inclination known as slope of a line $\boldsymbol{m = \tan\theta}$ $0 \le \theta \le 180°, \theta \ne 90°$

3. Slope of a line when two points are given

$A(x_1, y_1), B(x_2, y_2)$

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

Note that $x_1 \ne x_2$

- Slope of X-axis (or) slope of a line parallel to X-axis is '0' (zero)

  $m = \tan 0°$

  $m = 0$

- Slope of y-axis (or) slope of a line parallel to y-axis is **not defined**

  $m = \tan 90°$

  $m = \infty$

- $\tan 30° = 1/\sqrt{3}$

  $\tan 45° = 1$

  $\tan 60° = \sqrt{3}$

4. Slope of parallel lines

Two lines l_1 and l_2 are parallel if $\boldsymbol{m_1 = m_2}$

5. Slopes of perpendicular lines

Slope of a line is 'm' then its perpendicular line slope is $\frac{-1}{m}$

- l_1 is perpendicular to l_2 if and only if $\boldsymbol{m_1 \times m_2 = -1}$

Type: I [Problems based on slope]

Q.No: 1. (i) (ii), 2. (i) (ii), Example 5.8 (i) (ii), Example 5.9 (i) (ii) (iii), 3. (i) (ii), 7

1. What is the slope of a line whose inclination with positive direction of x-axis is (i) 90° (ii) 0°

(i) Slope $m = \tan\theta$

$m = \tan 90°$

$m = \infty$ (not defined)

(ii) Slope $m = \tan\theta$

$m = \tan 0°$

$m = 0$

2. What is the inclination of a line whose slope is (i) 0 (ii) 1

(i) Slope $m = \tan\theta$

$0 = \tan\theta$

$$\boxed{\theta = 0°}$$

(ii) Slope $m = \tan\theta$

$1 = \tan\theta$

$\theta = \tan^{-1}(1)$

$$\boxed{\theta = 45°}$$

Example 5.8

(i) What is the slope of a line whose inclination is 30°?

Slope $m = \tan\theta$

$m = \tan 30°$

$$\boxed{m = \frac{1}{\sqrt{3}}}$$

(ii) What is the inclination of a line whose slope is $\sqrt{3}$?

Given $m = \sqrt{3}$

$\tan\theta = \sqrt{3}$

$\theta = \tan^{-1}(\sqrt{3})$

$$\boxed{\theta = 60°}$$

Example 5.9

Find the slope of a line joining the given points (i) $\underset{x_1\ \ y_1}{(-6, 1)}$ and $\underset{x_2\ \ y_2}{(-3, 2)}$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2-1}{-3+6}$$

$$= \frac{1}{3}$$

(ii) $\left(-\frac{1}{3}, \frac{1}{2}\right)$ **and** $\left(-\frac{2}{7}, \frac{3}{7}\right)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\frac{3}{7} - \frac{1}{2}}{\frac{2}{7} + \frac{1}{3}}$$

$$= \frac{\frac{6-7}{14}}{\frac{6+7}{21}}$$

$$= \frac{\frac{-1}{14}}{\frac{13}{21}}$$

$$= \frac{-1}{14} \times \frac{21}{13}$$

$$m = \frac{-3}{26}$$

(iii) **(14, 10) and (14, $-b$)**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-6-10}{14-14}$$

$$= \frac{-16}{0}$$

$$m = \infty \text{ (not defined)}$$

3. Find the slope of a line joining the points (i) $(5, \sqrt{5})$ with the origin.

(i) $\begin{pmatrix} 5, & \sqrt{5} \\ x_1 & y_1 \end{pmatrix}$ and $\begin{pmatrix} 0, & 0 \\ x_2 & y_2 \end{pmatrix}$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{0 - \sqrt{5}}{0 - 5}$$

$$= \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

(ii) **$(\sin\theta, -\cos\theta)$ and $(-\sin\theta, \cos\theta)$**

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\cos\theta + \cos\theta}{-\sin\theta - \sin\theta}$$

$$= \frac{2\cos\theta}{-2\sin\theta}$$

$$m = -\cot\theta$$

Note

- $\frac{\sin\theta}{\cos\theta} = \tan\theta$
- $\frac{\cos\theta}{\sin\theta} = \cot\theta$

7. The line through the points $(-2, a)$ and (9,3) has slope $\frac{-1}{2}$. Find the value of 'a'.

$\underset{x_1\ \ \ y_1}{(-2,\ \ a)}$ and $\underset{x_2\ \ y_2}{(9\ \ \ 3)}$ $\quad m = \frac{-1}{2}$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{-1}{2} = \frac{3-a}{9+2}$$

$$\frac{-1}{2} = \frac{3-a}{11}$$

$$6 - 2a = -11$$

$$-2a = -11 - 6$$

$$-2a = -17$$

$$\boxed{a = 17/2}$$

Type: II Parallel, Perpendicular based sums.
Q.No: 5, Example 5.12 ,~ 6, 4, 8, Example 5.10, 5.11

5. Show that the given points are collinear: $(-3,-4)$, $(7, 2)$ and $(12, 5)$

Let $A(-3,-4)$ $B(7,2)$ and $C(12,5)$

Slope of AB	Slope of BC
$A\begin{matrix}(-3, & -4) \\ x_1 & y_1\end{matrix}$ $B\begin{matrix}(7, & 2) \\ x_1 & y_1\end{matrix}$	$B\begin{matrix}(7, & 2) \\ x_1 & y_1\end{matrix}$ $C\begin{matrix}(12, & 5) \\ x_2 & y_2\end{matrix}$
$m=\frac{y_2-y_1}{x_2-x_1}$	$m=\frac{y_2-y_1}{x_2-x_1}$
$=\frac{2+4}{7+3}$	$=\frac{5-2}{12-7}$
$=\frac{6}{10}$	$=\frac{3}{5}$
$m=\frac{3}{5}$	

slope of $AB =$ slope of BC and B is common point.

$\therefore$ Given points A, B, C are collinear.

Example 5.12

Show that the points $(-2,5)$, $(6,-1)$ and $(2,2)$ are collinear.

Let $A(-2,5)$, $B(6,-1)$ $C(2,2)$

Slope of AB	Slope of BC
$A\begin{matrix}(-2, & 5) \\ x_1 & y_1\end{matrix}$ $B\begin{matrix}(6, & -1) \\ x_2 & y_2\end{matrix}$	$B\begin{matrix}(6, & -1) \\ x_1 & y_1\end{matrix}$ $C\begin{matrix}(2, & 2) \\ x_2 & y_2\end{matrix}$
$m=\frac{y_2-y_1}{x_2-x_1}$	$m=\frac{y_2-y_1}{x_2-x_1}$
$=\frac{-1-5}{6+2}$	$=\frac{2+1}{2-6}$
$=\frac{-6}{8}$	$=\frac{3}{-4}$
$=\frac{-3}{4}$	$=-\frac{3}{4}$

Slope of $AB =$ slope of BC and B is common point.

$\therefore$ Given points A, B, C are collinear.

---

6. If the three points $(3,-1)$, $(a, 3)$ and $(1,-3)$ are collinear, find the value of 'a'.

Slope of AB	Slope of BC
$A\begin{matrix}(3, & -1) \\ x_1 & y_1\end{matrix}$ $B\begin{matrix}(a, & 3) \\ x_2 & y_2\end{matrix}$	$B\begin{matrix}(a, & 3) \\ x_1 & y_1\end{matrix}$ $C\begin{matrix}(1, & -3) \\ x_2 & y_2\end{matrix}$
$m=\frac{y_2-y_1}{x_2-x_1}$	$m=\frac{y_2-y_1}{x_2-x_1}$
$=\frac{3+1}{a-3}$	$=\frac{-3-3}{1-a}$
$=\frac{4}{a-3}$	$=\frac{-6}{1-a}$

Given points are collinear

$\therefore$ slope of $AB =$ slope of BC

$$\frac{4}{a-3}=\frac{-6}{1-a}$$

$$4(1-a)=-6(a-3)$$

$$4-4a=-6a+18$$

$$-4a+6a=18-4$$

$$2a=14$$

$$a=14/2$$

$$\boxed{a=7}$$

---

4. What is the slope of a line perpendicular to the line joining $A(5,1)$ and P where P where P is the mid-point of the segment joining $(4,2)$ and $(-6,4)$.

Let $B(4,2)$ and $C(-6,4)$

$P =$ mid point of BC

$$=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$=\left(\frac{4-6}{2},\frac{2+4}{2}\right)$$

$$=\left(\frac{-2}{2},\frac{6}{2}\right)$$

$$=(-1,3)$$

Slope of AP

$$A\begin{pmatrix}(5, & 1)\\ x_1 & y_1\end{pmatrix} P\begin{pmatrix}(-1 & 3)\\ x_2 & y_2\end{pmatrix}$$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$=\frac{3-1}{-1-5}$$

$$=\frac{2}{-6}$$

$$\boxed{m=\frac{-1}{3}}$$

Slope of perpendicular $=\dfrac{-1}{m}$
$= 3$

---

8. The line through the points $(-2, 6)$ and (4,8) is perpendicular to the line through the points (8, 12) and (x,24). Find the value of x.

Line 1

$A\ \begin{matrix}(-2, & 6)\\ x_1 & y_1\end{matrix}\ B\ \begin{matrix}(4, & 8)\\ x_2 & y_2\end{matrix}$

Slope of $AB=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{8-6}{4+2}$

$=\dfrac{2}{6}$

$m_1=\dfrac{1}{3}$

Line 2

$C\ \begin{matrix}(8, & 12)\\ x_1 & y_1\end{matrix}\ D\ \begin{matrix}(x, & 24)\\ x_2 & y_2\end{matrix}$

Slope of $CD=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{24-12}{x-8}$

$m_2=\dfrac{12}{x-8}$

Given line are perpendicular

$$\therefore m_1\times m_2=-1$$

$$\frac{1}{3}\times\frac{12}{x-8}=-1$$

$$\frac{4}{x-8}=-1$$

$$-x+8=4$$

$$-x=4-8$$

$$-x=-4$$

$$\boxed{x=4}$$

Example 5.10

The line 'r' passes through the points* $(-2, 2)$ and $(5, 8)$ *and the line 's' passes through the points* $(-8, 7)$ *and* $(-2, 0)$*. Is the line r perpendicular to S?

Line r

$A\ \begin{matrix}(-2, & 2)\\ x_1 & y_1\end{matrix}\ B\ \begin{matrix}(5, & 8)\\ x_2 & y_2\end{matrix}$

Slope of $AB=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{8-2}{5+2}$

$m_1=\dfrac{6}{7}$

Line s

$C\ \begin{matrix}(-8, & 7)\\ x_1 & y_1\end{matrix}\ D\ \begin{matrix}(-2, & 0)\\ x_2 & y_2\end{matrix}$

Slope of $CD=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{0-7}{-2+8}$

$m_2=\dfrac{-7}{6}$

Here $m_1\times m_2=\dfrac{6}{7}\times\dfrac{-7}{6}$

$=-1$

Hence given lines are perpendicular.

Example 5.11

The line 'P' passes through the points* $(3, -2), (12, 4)$ *and the line 'q' passes through the points* $(6, -2)$ *and* $(12, 2)$*. Is 'P' parallel to q.

Line p

$A\ \begin{matrix}(3, & -2)\\ x_1 & y_1\end{matrix}\ B\ \begin{matrix}(12, & 4)\\ x_2 & y_2\end{matrix}$

Slope of $AB=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{4+2}{12-3}$

$=\dfrac{6}{9}$

$m_1=\dfrac{2}{3}$

Line q

$C\ \begin{matrix}(6, & -2)\\ x_1 & y_1\end{matrix}\ D\ \begin{matrix}(12, & 2)\\ x_2 & y_2\end{matrix}$

Slope of $CD=\dfrac{y_2-y_1}{x_2-x_1}$

$=\dfrac{2+2}{12-5}$

$=\dfrac{4}{6}$

$m_2=\dfrac{2}{3}$

Here

Slope of line 'p' = slope of line 'q'. Hence given lines are parallel.

Type: III Geometrical Application sums - based an slope concept.

Q.No: 10, 11, 13, 12, Example 5.13, Example 5.15, 9. (i) (ii), 14, Example 5.14, 5.16

10. Show that the given points form a parallelogram $A\,(2.5, 3.5)$ $B\,(10, -4)$ $C\,(2.5, -2.5)$ and $D\,(-5, 5)$

Slope of AB

$A\underset{x_1\ \ y_1}{(2.5,\ 3.5)}$ $B\underset{x_2\ \ y_2}{(10,\ -4)}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4-3.5}{10-2.5} = \frac{-7.5}{7.5} = -1$$

Slope of CD

$C\underset{x_1\ \ y_1}{(2.5,\ -2.5)}$ $D\underset{x_2\ \ y_2}{(-5,\ 5)}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2.5}{-5-2.5} = \frac{7.5}{-7.5} = -1$$

Slope of BC

$B\underset{x_1\ \ y_1}{(10,\ -4)}$ $C\underset{x_2\ \ y_2}{(2.5,\ -2.5)}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2.5+4}{2.5-10} = \frac{1.5}{7.5} = -\frac{1}{5}$$

Slope of AD

$A\underset{x_1\ \ y_1}{(2.5,\ 3.5)}$ $D\underset{x_2\ \ y_2}{(-5,\ 5)}$

$$m = \frac{5-3.5}{-5-2.5} = \frac{1.5}{-7.5} = \frac{-1}{5}$$

Here slope of $AB =$ slope of CD

$$\therefore AB \parallel CD$$

Slope of $BC =$ slope of AD

$$\therefore BC \parallel AD$$

Since opposite sides are parallel, Given points are form a parallelogram.

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11. If the points $A\,(2, 2)$, $B\,(-2, -3)$ $C\,(1, -3)$ and $D\,(x, y)$ form a parallelogram then find the values of x and y

Slope of AB

$A\underset{x_1\ \ y_1}{(2,\ 2)}$ $B\underset{x_2\ \ y_2}{(-2,\ -3)}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-2}{-2-2} = \frac{-5}{-4} = \frac{5}{4}$$

Slope of CD

$C\underset{x_1\ \ y_1}{(1,\ -3)}$ $D\underset{x_2\ \ y_2}{(x,\ y)}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y+3}{x-1}$$

Given $ABCD$ is a parallelogram

$\therefore$ slope of $AB =$ slope of CD

$$\frac{5}{4} = \frac{y+3}{x-1}$$

$$5x - 5 = 4y + 12$$

$$5x - 4y = 12 + 5$$

$$5x - 4y = 17 \quad \text{...(1)}$$

Slope of BC

$B\underset{x_1\ \ y_1}{(-2,\ -3)}$ $C\underset{x_2\ \ y_2}{(1,\ -3)}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3+3}{1+2} = \frac{0}{3} = 0$$

Slope of AD

$A\underset{x_1\ \ y_1}{(2\ \ 2)}$ $D\underset{x_2\ \ y_2}{(x,\ y)}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y-2}{x-2}$$

Given $ABCD$ is a parallelogram

$\therefore$ slope of $BC =$ slope of AD

$$0 = \frac{y-2}{x-2}$$

$$y - 2 = 0$$

$$\boxed{y = 2}$$

Put $y = 2$ in (1)

$$5x - 4\,(2) = 17$$

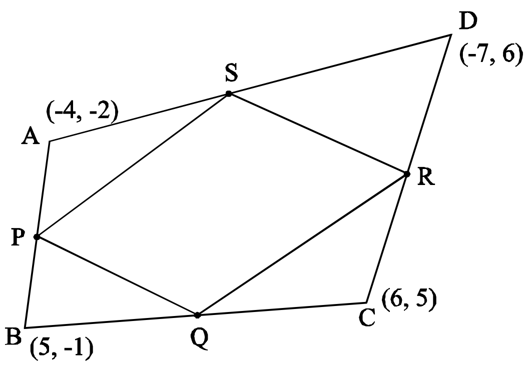
$$5x = 17 + 8$$

$$5x = 25$$

$$x = \frac{25}{5}$$

$$\boxed{x = 5}$$

13. A quadrilateral has vertices $A(-4,-2)$, $B(5,-1)$, $C(6,5)$ and $D(-7,6)$ show that the midpoints of its sides form a parallelogram.



- P = midpoint of AB

$$= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$= \left(\frac{-4+5}{2}, \frac{-2-1}{2}\right)$$

$$= \left(\frac{1}{2}, \frac{-3}{2}\right)$$

- Q = midpoint of BC

$$= \left(\frac{5+6}{2}, \frac{-1+5}{2}\right)$$

$$= \left(\frac{11}{2}, 2\right)$$

- R = midpoint of CD

$$= \left(\frac{6-7}{2}, \frac{5+6}{2}\right)$$

$$= \left(\frac{-1}{2}, \frac{11}{2}\right)$$

- S = midpoint of AD

$$= \left(\frac{-4-7}{2}, \frac{-2+6}{2}\right)$$

$$= \left(\frac{-11}{2}, 2\right)$$

Slope of PQ

$P\left(\frac{1}{2}, \frac{-3}{2}\right) Q\left(\frac{11}{2}, 2\right)$

$$m = \frac{y_2-y_1}{x_2-x_1}$$

$$m = \frac{2+3/2}{\frac{11}{2}-\frac{1}{2}}$$

$$= \frac{\frac{7}{2}}{\frac{10}{2}}$$

$$= \frac{7}{10}$$

Slope of PS

$p\left(\frac{1}{2}, \frac{-3}{2}\right) s\left(\frac{-11}{2}, 2\right)$

$$m = \frac{2+3/2}{\frac{-11}{2}-1/2}$$

$$= \frac{7/2}{-12/2} = \frac{-7}{12}$$

Slope of SR

$S\left(\frac{-11}{2}, 2\right) R\left(\frac{-1}{2}, \frac{11}{2}\right)$

$$m = \frac{\frac{11}{2}-2}{\frac{-1}{2}+\frac{11}{2}}$$

$$= \frac{\frac{7}{2}}{\frac{10}{2}}$$

$$= \frac{7}{10}$$

Slope of SR

$Q\left(\frac{11}{2}, 2\right) R\left(\frac{-1}{2}, \frac{11}{2}\right)$

$$m = \frac{\frac{11}{2}-2}{\frac{-1}{2}-\frac{-11}{2}}$$

$$= \frac{\frac{7}{2}}{\frac{-12}{2}}$$

$$= \frac{-7}{12}$$

Here slope of PQ = slope of SR

$$\therefore PQ \parallel SR$$

Slope of PS = slope of QR

$$PS \parallel QR$$

Since opposite sides are parallel $PQRS$ is a parallelogram.

---

12. Let $A(3,-4)$ $B(9,-4)$ $C(5,-7)$ and $D(7,-7)$. Show that $ABCD$ is a trapezium.

Slope of AB

$A(3,-4)$ $B(9,-4)$

$$m = \frac{y_2-y_1}{x_2-x_1}$$

$$m = \frac{-4+4}{9-3}$$

$$= \frac{0}{6}$$

$$= 0$$

Slope of CD

$C(5,-7)$ $D(7,-7)$

$$m = \frac{-7+7}{7-5}$$

$$= \frac{0}{2}$$

$$= 0$$

Slope of BC

$A\,(9, -4)\;\; C\,(5, -7)$

$$m = \frac{-7+4}{5-9}$$

$$= \frac{-3}{-4}$$

$$= \frac{3}{4}$$

Slope of AD

$A\,(3, -4)\;\; D\,(7, -7)$

$$m = \frac{-7+4}{7-3}$$

$$= \frac{-3}{4}$$

Here slope of $AB =$ slope of CD

$$\therefore AB \parallel CD$$

but slope of $BC \neq$ slope of AD

BC is not parallel to AD

Hence $ABCD$ is a trapezium.

Example 5.13

$A\,(1, -2)$ $B\,(6, -2)$ $C\,(5, 1)$ and $D\,(2, 1)$ be four points (i) Find the slope of the line segment (a) AB (b) CD (ii) Find the slope of line segment (a) BC (b) AD (iii) What can you deduce from your answer.

(i) (a) Slope of $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{-2+2}{6-1}$$

$$= \frac{0}{5}$$

$$= 0$$

(b) Slope of $CD = \dfrac{1-1}{2-5}$

$$= \frac{0}{-3}$$

$$= 0$$

(ii) (a) Slope of $BC = \dfrac{1+2}{5-6}$

$$= \frac{3}{-1}$$

$$= -3$$

(b) Slope of $AD = \dfrac{1+2}{2-1}$

$$= \frac{3}{1}$$

$$= 3$$

(iii) Slope of $AB =$ slope of CD

$$\therefore AB \parallel CD$$

Slope of $BC \neq$ slope of AD

$\therefore$ BC is not parallel to AD

Hence $ABCD$ is a trapezium.

Example 5.15

Without using Pythagoras theorem, show that the vertices $(1, -4), (2, -3)$ and $(4 - 7)$ form a right angled triangle.

Let the vertices of triangle

$A\,(1, -4)$ $B\,(2, -3)$ and $C\,(4 - 7)$

Slope of AB

$A \begin{array}{cc} (1, & -4) \\ x_1 & y_1 \end{array} \quad B \begin{array}{cc} (2, & -3) \\ x_2 & y_2 \end{array}$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-3+4}{2-1}$$

$$= \frac{1}{1}$$

$$= 1$$

Slope of BC

$B\,(2, -3)\;\; C\,(4, -7)$

$$m = \frac{-7+3}{4-2}$$

$$= \frac{-4}{2}$$

$$= -2$$

Slope of AC

$A\,(1, -4)$ $C\,(4 - 7)$

$$m = \frac{-7+4}{4-1}$$

$$= \frac{-3}{3}$$

$$= -1$$

Here slope of $AB \times$ slope of $AC =$

$$1 \times (-1) = -1$$

$\therefore AB$ is perpendicular to AC

i.e $\angle A = 90°$

Hence ΔABC is a right angled triangle.

---

9. Show that the given vertices form a right angled triangle and check whether its satisfies Pythagoras theorem

(i) A (1, – 4) B (2, – 3) and C (4, – 7)

$$A \begin{matrix} (1, & -4) \\ x_1 & y_1 \end{matrix} \quad B \begin{matrix} (2, & -3) \\ x_2 & y_2 \end{matrix}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3+4}{2-1} = \frac{1}{1} = 1$$

Length of AB

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(1)^2 + (1)^2}$$

$$= \sqrt{2}$$

Slope of BC

B (2, – 3) C (4, – 7)

$$m = \frac{-7+3}{4-2}$$

$$= \frac{-4}{2}$$

$$= -2$$

Length of BC

$$BC = \sqrt{(4-2)^2 + (-7+3)^2}$$

$$= \sqrt{(2)^2 + (-4)^2}$$

$$= \sqrt{4+16}$$

$$= \sqrt{20}$$

Slope of AC

A (1, – 4) C (4, – 7)

$$m = \frac{-7+4}{4-1}$$

$$= \frac{-3}{3}$$

$$= -1$$

Length of AC

$$AC = \sqrt{(4-1)^2 + (-7+4)^2}$$

$$= \sqrt{(3)^2 + (-3)^2}$$

$$= \sqrt{9+9}$$

$$= \sqrt{18}$$

Here slope of $AB \times$ slope of AC

$$= 1 \times (-1)$$

$$= -1$$

$\therefore AB \perp AC$ and $\angle A = 90°$

ΔABC is a right angled triangle.

Also

Here $BC^2 = AB^2 + AC^2$

$$(\sqrt{20})^2 = (\sqrt{2})^2 + (\sqrt{18})^2$$

$$20 = 2 + 18$$

$$20 = 20 \text{ and } \angle A = 90°$$

It satisfies Pythagoras theorem.

(ii) L (0, 5) M (9, 12) and N (3, 14)

Slope of LM

L (0, 5) M (9, 12)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{12-5}{9-0}$$

$$= \frac{7}{9}$$

Length of LM

L (0, 5) M (9, 12)

$$LM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(9-0)^2 + (12-5)^2}$$

$$= \sqrt{9^2 + 7^2}$$

$$= \sqrt{81+49}$$

$$= \sqrt{130}$$

Slope of MN

M (9, 12) N (3, 14)

$$m = \frac{14-12}{3-9}$$

$$= \frac{2}{-6}$$

$$= \frac{-1}{3}$$

Length of MN

$$MN = \sqrt{(3-9)^2 + (14-12)^2}$$

$$= \sqrt{(-6)^2 + (2)^2}$$

$$= \sqrt{36+4}$$

$$= \sqrt{40}$$

Slope of LN

L (0, 5) N (3, 14)

$$m = \frac{14-5}{3-0}$$

$$= \frac{9}{3}$$

$$= 3$$

Length of LN

$$LN = \sqrt{(3-0)^2 + (14-5)^2}$$

$$= \sqrt{3^2 + 9^2}$$

$$= \sqrt{9+81}$$

$$= \sqrt{90}$$

Here slope of $MN \times$ slope of LN

$$= \frac{-1}{3} \times 3$$

$$= -1$$

Hence $MN \perp LN$ and $\angle LN = 90°$

$\therefore \Delta\, LMN$ is a right angled triangle.

Also,

$$LM^2 = MN^2 + LN^2$$

$$(\sqrt{130})^2 = (\sqrt{40})^2 + (\sqrt{90})^2$$

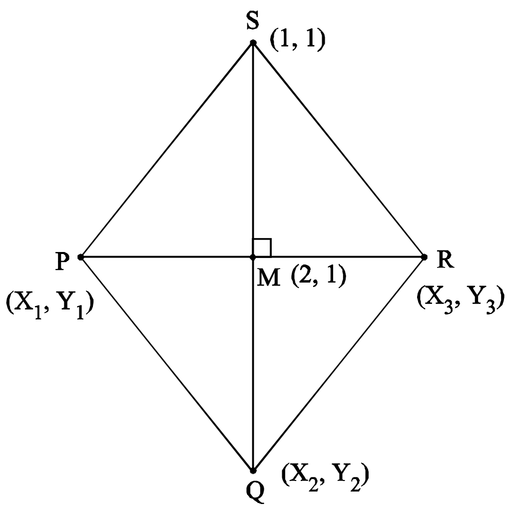
$$130 = 40 + 90$$

$$130 = 130 \text{ and } LN = 90°$$

It satisfies pythagoras theorem.

---

14. *PQRS* is a rhombus. Its diagonals *PR* and *QS* intersect at the point *M* and satisfy $QS = 2PR$. If the co-ordinates of *S* and *M* are (1,1) and (2, – 1) respectively, find the co-ordinates of *P*.



In a Rhombus diagonals bisect each other at right angle.

$$\therefore M = \text{midpoint of } QS$$

$$(2, -1) = \left(\frac{1+x_2}{2}, \frac{1+y_2}{2}\right)$$

$$2 = \frac{1+x_2}{2}$$
$$1 + x_2 = 4$$
$$x_2 = 4 - 1$$
$$\boldsymbol{x_2 = 3}$$

$$-1 = \frac{1+y_2}{2}$$
$$1 + y_2 = -2$$
$$y_2 = -2 - 1$$
$$\boldsymbol{y_2 = -3}$$

$\therefore$ vertex $Q\,(3, -3)$

Given

$$QS = 2PR$$
$$QS^2 = 4PR^2$$
$$(3-1)^2 + (-3-1)^2 = 4PR^2$$
$$4 = 16 = 4PR^2$$
$$\frac{20}{4} = PR^2$$
$$5 = PR^2$$
$$PR = \sqrt{5}$$
$$\therefore PM = \frac{1}{2}\sqrt{5}$$

we have

$$SP^2 = QP^2$$
$$(x-1)^2 + (y-1)^2 = (x-3)^2 + (y+3)^2$$
$$x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 - bx + 9 + y^2 + 6y + 9$$
$$-2x - 2y + 2 = -6x + 6y + 18$$
$$4x - 8y = 16$$
$$x - 2y = 4$$
$$\boxed{\boldsymbol{x = 2y + 4}} \qquad \text{...(1)}$$

Also

$$PM^2 = \left(\frac{2}{2}\sqrt{5}\right)^2$$
$$(x-2)^2 + (y+1)^2 = \frac{5}{4}$$
$$x^2 - 4x + 4 + y^2 + 2y + 1 = 5/4$$
$$x^2 + y^2 - 4x + 2y + 5 = 5/4$$

Put $x = 2y + 4$

$$(2y+4)^2 + y^2 - 4\,(2y+4) + 2y + 5 = 5/4$$
$$4y^2 + 16y + 16 + y^2 - 8y - 16 + 2y + 5 = 5/4$$
$$5y^2 + ,10y + 5 = 5/4$$
$$20y^2 + 40y + 15 = 0$$
$$4y^2 + 8y + 3 = 0$$
$$(\,y + 3/2\,)\left(y + \frac{1}{2}\right) = 0$$
$$\boxed{\boldsymbol{y = -3/2 \quad -1/2}}$$

$$\begin{array}{cc} & 12 \\ 6 & 2 \\ \frac{6}{4} & \frac{2}{4} \\ \frac{3}{2} & \frac{1}{2} \end{array}$$

(1) $\Rightarrow$

$x = 2y + 4$	$x = 2y + 4$
$= 2\left(\frac{-3}{2}\right) + 4$	$= 2\left(\frac{-1}{2}\right) + 4$
$= -3 + 4$	$= -1 + 4$
$x = 1$	$x = 3$
$\therefore (1, -3/2)$	$(3, -1/2)$

$\therefore$ Co-ordinates of

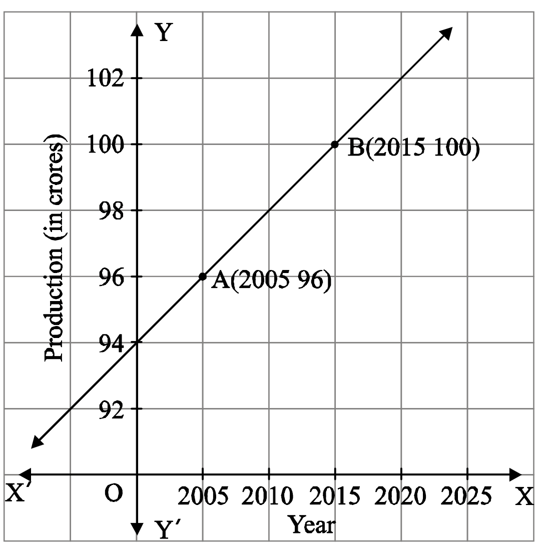
P are $(1, -3/2)$ and $(3, -1/2)$

Example 5.14

Consider the graph representing growth of population (in crores). Find the slope of the line AB and hence estimate the population in the year 2030?

Solution:

The points $A\,(2005, 96)$ and $B\,(2015, 100)$ are on the line AB.



Slope of $AB = \frac{100 - 96}{2015 - 2005} = \frac{4}{10} = \frac{2}{5}$

Let the growth of population in 2030 be k crores.

Assuming that the point $C\,(2030, k)$ is on AB,

we have, slope of $AC =$ slope of AB

$$\frac{k - 96}{2030 - 2005} = \frac{2}{5}$$

$$\frac{k - 96}{25} = \frac{2}{5}$$

$$k - 96 = 10$$

$$k = 10 + 96$$

$$k = 106$$

Hence the estimated population in $2030 = 106$ Crores.

Example 5.16

Prove analytically that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is equal to half of its length.

Solution:

Let $P\,(a, b)$ $Q\,(c, d)$ and $R\,(e, f)$ be the vertices of a triangle.

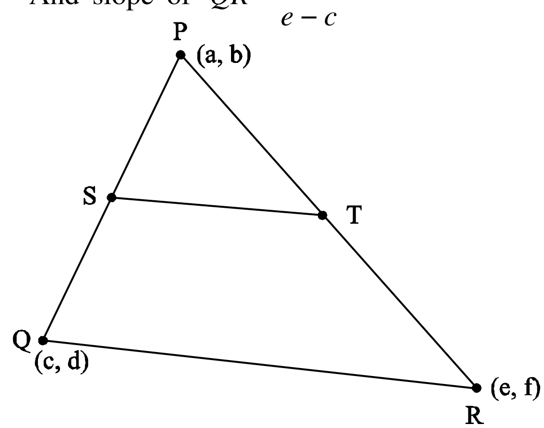
Let S be the mid-point of PQ and T be the mid-point of PR

Therefore,

$$S = \left(\frac{a+c}{2}, \frac{b+d}{2}\right) \text{ and } T = \left(\frac{a+e}{2}, \frac{b+f}{2}\right)$$

Now, slope of $ST = \dfrac{\frac{b+f}{2} - \frac{b+d}{2}}{\frac{a+e}{2} - \frac{a+c}{2}} = \dfrac{f-d}{e-c}$

And slope of $QR = \dfrac{f-d}{e-c}$



Therefore, ST is parallel to QR. (since their slopes are equal)

$$\text{Also, } ST = \sqrt{\left(\frac{a+e}{2} - \frac{a+c}{2}\right)^2 + \left(\frac{b+f}{2} - \frac{b+d}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(e-c)^2 + (f-d)^2}$$

$$ST = \frac{1}{2} QR$$

Thus ST is parallel to QR and half of it.

Note

- This example illustrates how a geometrical result can be proved using coordinate Geometry.

Exercise 5.3

Equation of a Straight line

I. Equation of co-ordinate axes

1. Equation of a line parallel to X-axis is $\boldsymbol{y = b}$

- If $b > 0$, then the line $y = b$ lies above the x-axis.
- If $b < 0$, then the line $y = b$ lies below the x-axis.
- If $b = 0$, then the line $y = b$ is the x-axis itself.

2. Equation of a line parallel to y-axis is $\boldsymbol{x = c}$

- If $c > 0$ then the line $x = c$ lies right to the side of the y-axis.
- If $c < 0$ then the line $x = c$ lies left to the side of the y-axis.
- If $c = 0$ then the line $x = c$ is the y-axis itself.

II. Equation of a line (Neither parallel nor perpendicular)

3. Slope - Intercept form

A line with slope 'm' and 'y' intercept 'c' then equation

$$\boxed{\boldsymbol{y = mx + c}}$$

- If a line with slope m, $m \neq 0$ makes x intercept d then the equation of the straight line $y = m(x - d)$
- $y = mx$ is the equation of a straight line with slope 'm' and passing through the origin.

4. Point - slope form

Equation of a line passing through a point (x_1, y_1) with slope 'm' is

$$\boxed{\boldsymbol{y - y_1 = m(x - x_1)}}$$

5. Two points form

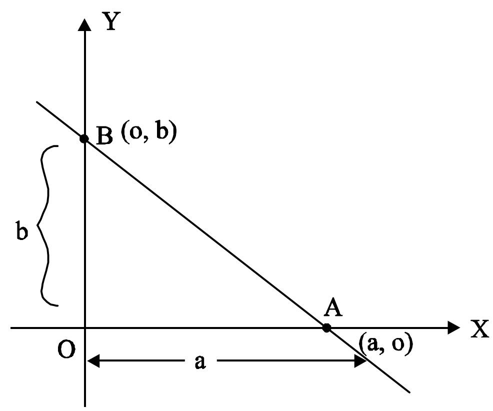
Equation of a line passing through two pints $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\boxed{\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}}$$

6. Intercept form

Equation of a line having x intercept 'a' and 'y' intercept 'b' is

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$



Type: I Equation of co-ordinate axes Example 5.17, 1

Example 5.17

Find the equation of the straight line passing through (5,7) and is (i) Parallel to x-axis (ii) Parallel to y-axis.

(i) Equation of a line parallel to x-axis is $y = b$
since it passes through (5,7), so $b = 7$ then required equation is $\boldsymbol{y = 7}$

(ii) Equation of a line parallel to y-axis is $x = a$
since it passes through (5,7), So, $a = 5$ then required equation is $\boldsymbol{x = 5}$

1. Find the equation of a straight line passing through the midpoint of a line segment joining the points (1, – 5) (4, 2) and parallel to (i) X-axis (ii) y-axis.

P = midpoint of $(1, -5)$ and $(4, 2)$

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$P = \left(\frac{1 + 4}{2}, \frac{-5 + 2}{2}\right)$$

$$= \left(\frac{5}{2}, \frac{-3}{2}\right)$$

(i) Equation of a line parallel to x-axis $y = b$

Since it passes through $\left(\frac{5}{2}, \frac{-3}{2}\right)$

$\therefore b = \frac{-3}{2}$, then required equation $y = \frac{-3}{2}$

$$\boxed{2y + 3 = 0}$$

(ii) Equation of a line parallel to y-axis $x = a$

since it passes through $\left(\frac{5}{2}, \frac{-3}{2}\right)$

$\therefore a = \frac{5}{2}$, then required.

equation $x = \frac{5}{2}$

$$\boxed{2x - 5 = 0}$$

Type: II Slope Intercept form $\boldsymbol{y = mx + c}$
Q.No: 3. Example 5.18 (i) (ii), 2, 4 Example 5.19 5.20, 5.

3. Find the equation of a line whose inclination is 30° and making an intercept – 3 on the y-axis.

Given $\theta = 30°$, y-intercept '$C = -3$'

Slope $m = \tan\theta$

$m = \tan 30°$

$m = 1/\sqrt{3}$

$\therefore$ **equation**

$$y = mx + c$$

$$y = \frac{1}{\sqrt{3}}x - 3$$

$$\sqrt{3}\, y = x - 3\sqrt{3}$$

$$\boxed{x - \sqrt{3}\, y - 3\sqrt{3} = 0}$$

Example 5.18

Find the equation of a straight line whose
(i) slope is 5 and y intercept is – 9.

Given slope $m = 5$, y-intercept 'c' $= -9$

Equation

$$y = mx + c$$

$$y = 5x - 9$$

$$\boxed{5x - y - 9 = 0}$$

(ii) Inclination is 45° and y-intercept is 11.

Given $\theta = 45°$; y-intercept 'c'$= 11$

slope $m = \tan 45°$

$m = 1$

$\therefore$ **equation**

$$y = mx + c$$

$$y = x + 11$$

$$\boxed{x - y + 11 = 0}$$

2. The equation of a straight line is $2(x-y)+5=0$. Find its slope, inclination and intercept of the y-axis.

Given

$$2(x-y)+5=0$$

$$2x-2y+5=0$$

$$2x+5=2y$$

$$\frac{2x+5}{2}=y$$

$$y=\frac{2x}{2}+\frac{5}{2}$$

$$y=x+\frac{5}{2}$$

comparing with

$y=mx+c$ we get

slope 'm' = 1 and y intercept **'c' = 5/2**

Here $m=1$

$\therefore \tan\theta=1$

$$\boxed{\theta=45°}$$

---

4. Find the slope and y-intercept of $\sqrt{3x}+(1-\sqrt{3})y=3$

Given

$$\sqrt{3}\,x+(1-\sqrt{3})\,y=3$$

$$(1-\sqrt{3})\,y=-\sqrt{3}\,x+3$$

$$y=\frac{-\sqrt{3}\,x}{1-\sqrt{3}}+\frac{3}{1-\sqrt{3}}$$

Comparing with $y=mx+c$

We get,

$$m=\frac{-\sqrt{3}}{1-\sqrt{3}}$$

$$=\frac{\sqrt{3}}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$=\frac{3+\sqrt{3}}{(\sqrt{3})^2-(1)^2}$$

$$=\frac{3+\sqrt{3}}{3-1}$$

$$\boxed{m=\frac{3+\sqrt{3}}{2}}$$

$$c=\frac{3}{1-\sqrt{3}}$$

$$=\frac{3}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$=\frac{3+3\sqrt{3}}{(1)^2-(\sqrt{3})^2}$$

$$=\frac{3+3\sqrt{3}}{1-3}$$

$$\boxed{c=\frac{3+3\sqrt{3}}{-2}}$$

---

5. Find the value of 'a', if the line through $(-2, 3)$ and $(8, 5)$ is perpendicular to $y=ax+2$

Line 1:

$(-2, 3)$ and $(8, 5)$

$$m=\frac{y_2-y_1}{x_2-x_1}$$

$$=\frac{5-3}{8+2}$$

$$=\frac{2}{10}$$

$$m_1=\frac{1}{5}$$

Line 2:

$$y=ax+2$$

comparing with $y=mx+c$

$$m=a$$

i.e $m_2=a$

Given lines are perpendicular

$\therefore m_1\times m_2=-1$

$$\frac{1}{5}\times a=-1$$

$$\boxed{a=-5}$$

Example 5.19

Calculate the slope and y intercept of the straight line $8x - 7y + 6 = 0$.

Given

$$8x - 7y + 6 = 0$$

$$8x + 6 = 7y$$

$$\frac{8x+6}{7} = y$$

$$y = \frac{8}{7}x + \frac{6}{7}$$

comparing with $y = mx + c$, we get

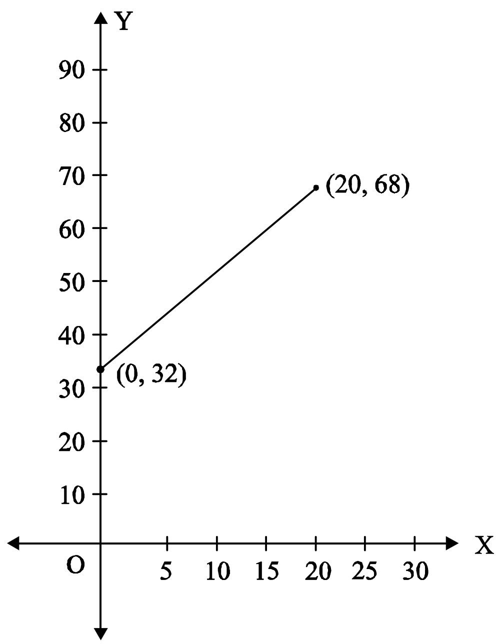
$$m = \frac{8}{7}, c = \frac{6}{7}$$

Example 5.20

The graph relates temperatures y (in Fahrenheit degree) to temperatures x (in celsius degree) (a) Find the slope and y-intercept (b) write an equation of the line (c) what is the mean temperature of the earth in Fahrenheit degree if its mean temperature is 25° celsius?

From the figure

Let $A(0, 32)$, $B(20, 68)$



(a) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{68 - 32}{20 - 0}$$

$$= \frac{36}{20} = \frac{9}{5}$$

$$m = 1.8$$

The line meet y axis at (0,32) so y-intercept 'c' is 32.

(b) Equation: $y = mx + c$

$$y = \frac{9}{5}x + 32$$

(c) Given $x = 25°$

$$\therefore y = \frac{9}{5}(25) + 32$$

$$y = 45 + 32$$

$$y = 77$$

$\therefore$ the mean temperature of the earth is 77°F.

Type: III Point, slope form
$y - y_1 = m(x - x_1)$

Q.No: Example 5.21, 10, 6, Example 5.22.

Example 5.21

Find the equation of a line passing through the point $(3, -4)$ and having slope $-5/7$

Given $m = \dfrac{-5}{7}$, $(3, -4)$

Equation

$$y - y_1 = m(x - x_1)$$

$$y + 4 = \frac{-5}{7}(x - 3)$$

$$7y + 28 = -5x + 15$$

$$5x + 7y + 28 - 15 = 0$$

$$5x + 7y + 13 = 0$$

10. Find the equation of a straight line which has slope $\frac{-5}{4}$ and passing through the point $(-1, 2)$

Given: $m = \frac{-5}{4}, (-1, 2)$

Equation

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{-5}{4}(x + 1)$$

$$4y - 8 = -5x - 5$$

$$5x + 4y - 8 + 5 = 0$$

$$5x + 4y - 3 = 0$$

---

6. The hill is in the form of a triangle has its foot at (19,3). The inclination of the hill to the ground is 45°. Find the equation of the hill joining the foot and top.

Given $\theta = 45°$

$$m = \tan 45°$$

$m = 1$ and point $(19, 3)$

Equation

$$y - y_1 = m(x - x_1)$$

$$y - 3 = 1(x - 19)$$

$$y - 3 = x - 19$$

$$x - 19 - y + 3 = 0$$

$$x - y - 16 = 0$$

$\therefore$ equation of the hill joining the foot and top is $x - y - 16 = 0$

Example 5.22

Find the equation of a line passing through the point A (1, 4) and perpendicular to the line joining the points (2,5) and (4,7)

Let given points $A(1, 4)$ $B(2, 5)$ and $C(4, 7)$

Slope of $BC = \frac{y_2 - y_2}{x_2 - x_1}$

$$= \frac{7 - 5}{4 - 2}$$

$$= \frac{2}{2}$$

$$m = 1$$

Perpendicular line slope is $\frac{-1}{m}$

$$\therefore m = -1$$

Now equation of the perpendicular line through $A(1, 4)$

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -1(x - 1)$$

$$y - 4 = -x + 1$$

$$x - 1 + y - 4 = 0$$

$$\boxed{x + y - 5 = 0}$$

Type: IV Two points form.

Q.No: 7. (ii) (i), 8, Example 5.23, 5.24, 9.

7. Find the equation of a line through the given pair of points (ii) (2,3) and (−7, −1)

$A \begin{matrix} (2, & 3) \\ x_1 & y_1 \end{matrix}$ and $B \begin{matrix} (-7, & -1) \\ x_2 & y_2 \end{matrix}$

Equation

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y -}{-1 - 3} = \frac{x - 2}{-7 - 2}$$

$$\frac{y - 3}{-4} = \frac{x - 12}{-9}$$

$$4(x - 2) = 9(y - 3)$$

$$4x - 8 = 9y - 27$$

$$4x - 8 - 9y + 27 = 0$$

$$4x - 9y + 19 = 0$$

(i) $\left(2, \frac{2}{3}\right)$ **and** $\left(\frac{-1}{2}, -2\right)$

Equation

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-2/3}{-2-2/3} = \frac{x-2}{\frac{-1}{2}-2}$$

$$\frac{\frac{3y-2}{3}}{\frac{-6-2}{3}} = \frac{x-2}{\frac{-1-4}{2}}$$

$$\frac{3y-2}{-8} = \frac{2(x-2)}{-5}$$

$$16(x-2) = 5(3y-2)$$

$$16x - 32 = 15y - 10$$

$$16x - 32 - 15y + 10 = 0$$

$$\boxed{\mathbf{16x - 15y - 22 = 0}}$$

---

8. A cat is located at the point $(-6, -4)$ in xy plane. A bottle of milk is kept at (5,11). The cat wishes to consume the milk travelling through shortest possible distance. Find the equation of the path it needs to take its milk.

Let $A\begin{pmatrix} -6, & -4 \\ x_1 & y_1 \end{pmatrix}$ and $B\begin{pmatrix} 5, & 11 \\ x_2 & y_2 \end{pmatrix}$

equation $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$

$$\frac{y+4}{11+4} = \frac{x+6}{5+6}$$

$$\frac{y+4}{15} = \frac{x+6}{11}$$

$$15(x+6) = 11(y+4)$$

$$15x + 90 = 11y + 44$$

$$15x - 11y + 90 - 44 = 0$$

$$15x - 11y + 46 = 0$$

$\therefore$ equation of the path to take its milk is $15x - 11y + 46 = 0$

Example 5.23

***Find the equation of a straight line passing through* (5, – 3) and (7 – 4)**

Let $A\begin{pmatrix} 5, & -3 \\ x_1 & y_1 \end{pmatrix}$ and $B\begin{pmatrix} 7, & -4 \\ x_2 & y_2 \end{pmatrix}$

Equation

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y+3}{-4+3} = \frac{x-5}{7-5}$$

$$\frac{y+3}{-1} = \frac{x-5}{2}$$

$$2(y+3) = -1(x-5)$$

$$2y + 6 = -x + 5$$

$$2y + 6 + x - 5 = 0$$

$$\boxed{\mathbf{x + 2y + 1 = 0}}$$

Example 5.24

Two buildings of different heights are located t opposite sides of each other. If a heavy rod is attached joining the terrace of the building from (6,10) to (14,12). Find the equation of the rod joining the buildings?

Let $A\begin{pmatrix} 6, & 10 \\ x_1 & y_1 \end{pmatrix}$ and $B\begin{pmatrix} 14, & 12 \\ x_2 & y_2 \end{pmatrix}$

Equation

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-10}{12-10} = \frac{x-6}{14-6}$$

$$\frac{y-10}{2} = \frac{x-6}{8}$$

$$\frac{y-10}{1} = \frac{x-6}{4}$$

$$x - 6 = 4y - 40$$

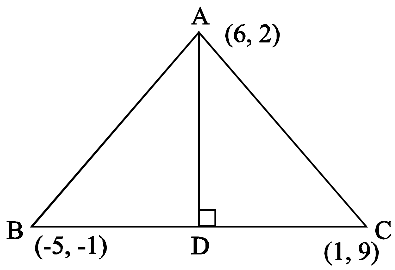
$$x - 6 - 4y + 40 = 0$$

$$x - 4y + 34 = 0$$

$\therefore$ equation of the rod is $x - 4y + 34 = 0$

9. Find the equation of the median and altitude of ΔABC through A where the vertices are A (6, 2), B (– 5, – 1) and C (1, 9)

(i) Equation of Median



D = Midpoint of BC

$$= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$= \left(\frac{-5+1}{2}, \frac{-1+9}{2}\right)$$

$$= \left(\frac{-4}{2}, \frac{8}{2}\right)$$

$$= (-2, 4)$$

Equation of Median AD

$$A\begin{pmatrix} 6, & 2 \\ x_1 & y_1 \end{pmatrix} \text{and } D\begin{pmatrix} -2, & 4 \\ x_2 & y_2 \end{pmatrix}$$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-2}{4-2} = \frac{x-6}{-2-6}$$

$$\frac{y-2}{2} = \frac{x-6}{-8}$$

$$\frac{y-2}{1} = \frac{x-6}{-4}$$

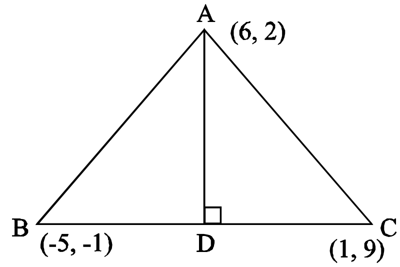
$$x-6 = -4(y-2)$$

$$x-6 = -4y+8$$

$$x-6+4y-8 = 0$$

$$\boxed{\mathbf{x+4y-14=0}}$$

(ii) Equation of attitude



$$\text{Slope of } BC = \frac{y_2-y_1}{x_2-x_1}$$

$$= \frac{9+1}{1+5}$$

$$= \frac{10}{6}$$

$$= \frac{5}{3}$$

Slope of AD = perpendicular slope of BC

$m = -3/5$

∴ Equation of attitude AD

A (6, 2) $m = -3/5$

$$y - y_1 = m(x-x_1)$$

$$y-2 = \frac{-3}{5}(x-6)$$

$$5y-10 = -3x+18$$

$$5y-10+3x-18 = 0$$

$$\boxed{\mathbf{3x+5y-28=0}}$$

Type: V Intercepts form $\frac{x}{a}+\frac{y}{b}=1$

QNo: 12. (i) (ii), 13, (i) (ii), Example 5.26, 14. (i) (ii), Example 5.25, 5.28.

12. Find the equation of a line whose intercepts on the x and y axes are given below.

(i) 4, – 6

Given

x intercept 'a' = 4

y intercept 'b' = – 6

Equation

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\frac{x}{4}+\frac{y}{-6}=1$$

$$\frac{x}{4}-\frac{y}{6}=1$$

$$\frac{3x-2y}{12}=1$$

$$3x-2y=12$$

$$\boxed{\mathbf{3x-2y-12=0}}$$

(ii) $\mathbf{-5, \frac{3}{4}}$

Given

x intercept 'a' $=-5$

y intercept 'b' $=3/4$

Equation

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\frac{x}{-5}+\frac{y}{3/4}=1$$

$$\frac{-x}{5}+\frac{4y}{3}=1$$

$$\frac{-3x+20y}{15}=1$$

$$-3x+20y=15$$

$$\boxed{\mathbf{3x-20y+15=0}}$$

13. Find the intercepts made by the following lines on the co-ordinate axes.

(i) $\mathbf{3x-2y-6=0}$

$$3x-2y-6=0$$

$$3x-2y-6=0$$

$$3x-2y=6$$

divide by 6

$$\frac{3x}{6}+\frac{(-2y)}{6}=\frac{6}{6}$$

$$\frac{x}{2}+\frac{y}{-3}=1$$

Comparing with $\frac{x}{a}+\frac{y}{b}=1$, we get

$$\boxed{\mathbf{a=2, b=-3}}$$

(ii) $\mathbf{4x+3y+12=0}$

$4x+3y+12=0$ (Another method)

***x* intercept**	***y* intercept**
Put $y=0$	Put $\mathbf{x=0}$
$4x+12=0$	$3y+12=0$
$4x=-12$	$3y=-12$
$x=-12/4$	$y=-12/3$
$x=-3$	$y=-4$
$\therefore$ x intercept $\mathbf{a=-3}$	y intercept $\mathbf{b=-4}$

Example 5.26

Find the intercepts made by the line $\mathbf{4x-9y+36=0}$ on the co-ordinate axes.

$$4x-9y+36=0$$

$$4x-9y=-36$$

divide by -36

$$\frac{4x}{-36}-\frac{9y}{-36}=\frac{-36}{-36}$$

$$\frac{x}{-9}+\frac{y}{4}=1$$

$\therefore$ x intercept 'a' $=-9$

y intercept 'b' $=4$

14. Find the equation of a straight line

(i) Passing through $(1,-4)$ and has intercepts which are in the ratio 2:5

Given:

$$a:b=2:5$$

$$\frac{a}{b}=\frac{2}{5}$$

$$2b = 5a$$

$$\boxed{b = \frac{5a}{2}}$$

Equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{a} + \frac{y}{\frac{5a}{2}} = 1$$

$$\frac{x}{a} + \frac{2y}{5a} = 1$$

$$\frac{5x + 2y}{5a} = 1$$

$$5x + 2y = 5a \qquad ...(1)$$

eqn (1) passes through $(1, -4)$

$$5(1) + 2(-4) = 5a$$

$$5 - 8 = 5a$$

$$\boxed{-3 = 5a}$$

$\therefore$ Required equation

$$(1) \Rightarrow 5x + 2y = -3$$

$$\boxed{5x + 2y + 3 = 0}$$

(ii) Passing through $(8, 4)$ and making equal intercepts on the co-ordinate axes.

Given;

$$a = b$$

Equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

Put $a = b$

$$\frac{x}{a} + \frac{y}{a} = 1$$

$$\frac{x + y}{a} = 1$$

$$x + y = a \qquad ...(1)$$

eqn (1) passing through (4) point $(-8, 4)$

$$\therefore -8 + 4 = a$$

$$\boxed{-4 = a}$$

$\therefore$ Required equation

$$(1) \Rightarrow x + y = -4$$

$$\boxed{x + y + 4 = 0}$$

Example 5.25

Find the equation of a line which passes through (5,7) and makes intercepts on the axes equal in magnitude but opposite in sign.

Given

$$a = -b$$

Equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

put $a = -b$

$$\frac{x}{-b} + \frac{y}{b} = 1$$

$$\frac{-x + y}{b} = 1$$

$$-x + y = b \qquad ...(1)$$

(5,7) passing through equ (1)

$$-5 + 7 = b$$

$$\boxed{2 = b}$$

$$(1) \Rightarrow -x + y = 2$$

$$\boxed{x - y + 2 = 0}$$

Example 5.28

A line makes positive intercepts on co-ordinate axes whose sum is 7 and it passes through $(-3, 8)$. Find its equation.

Given

Sum of intercepts $= 7$

$$a + b = 7$$

$$\boxed{b = 7 - a}$$

Equation

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\frac{x}{a}+\frac{y}{7-a}=1$$

$$\frac{x(7-a)+ay}{a(7-a)}=1$$

$$x(7-a)+ay=a(7-a) \qquad ...(1)$$

equ (1) passing through $(-3, 8)$

$(1) \Rightarrow -3(7-a)+a(8)=a(7-a)$

$$-21+3a+8a=7a-a^2$$

$$-21+11a=7a-a^2$$

$$a^2-7a+11a-21=0$$

$$a^2+4a-21=0$$

$$(a-3)(a+7)=0$$

$a-3=0$ $\quad$ $a+7=0$

$\boldsymbol{a=3}$ $\quad$ $\boldsymbol{a=-7}$ negative

(not ,possible since given positive intercept)

$\therefore$ **Required equation**

put $a=3$ in (1)

$x(7-3)+3y=3(7-3)$

$$4x+3y=3(4)$$

$$\boxed{\mathbf{4x+3y-12=0}}$$

Type: VI Application of equation of straight line sums.

QNo: 11. Example 5.27, 5.29.

11. You are downloading a song. The percent y (in decimal form) of mega bytes remaining to get downloaded in x seconds is given by $y=-0.1x+1$

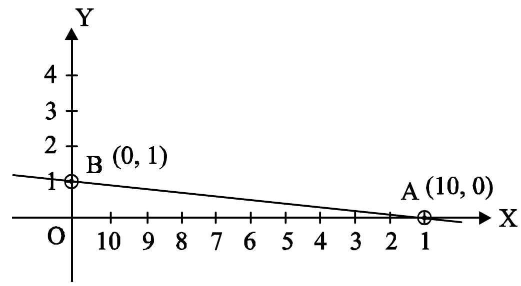
(i) Graph the equation.

(ii) Find the total MB of the song.

(iii) after howmany seconds will 75% of the songs gets downloaded.

(iv) after how many seconds the song will be downloaded completely?.

(i) Graph of $y=-0.1x+1$



Put $x=0$

We get $y=1$

$B(0, 1)$

Put $y=0$

we get $x=10$

$A(10, 0)$

(ii) $y=-0.1x+1$

when $x=0, y=1$

$\therefore$ Total MB of the song $=1\,MB$

(iii) 75% of MB downloaded

$\therefore$ 25% of MB to be downloaded

Put $y=0.25$ in $y=-0.1x+1$

$0.25=-0.1x+1$

$0.1x=1-0.25$

$0.1x=0.75$

$$x=\frac{0.75}{0.1}\times\frac{10}{10}$$

$$x=7.5$$

$\therefore$ Required time $=7.5$ sec

(iv) Time when songs completely downloaded

Put $y=0$ in $y=-0.1x+1$

$0=-0.1x+1$

$0.1x=1$

$$x=\frac{1}{0.1}\times\frac{10}{10}$$

$$x=10$$

$\therefore$ songs will be downloaded completely after 10 sec.

Type: I Slope based sums

QNo: 1. (i) (ii) Example 5.30, 5.31 (i) (ii) 2. (i) (ii), 3. (i) (ii), Example 5.32, 5.33, 4,

1. Find the slope of the following straight lines (i) $5y - 3 = 0$ (ii) $7x - \frac{3}{17} = 0$

(i) $5y - 3 = 0$

$$m = -\frac{\text{co}-\text{efficient of } x}{\text{co}-\text{efficient of } y}$$

$$= \frac{-0}{5}$$

$$m = 0$$

(ii) $7x - 3/17 = 0$

$$m = -\frac{\text{co}-\text{efficient of } x}{\text{co}-\text{efficient of } y}$$

$$= -7/0$$

$$m = \infty \text{ (undefined)}$$

Example 5.30

Fine the slope of the straight line $6x + 8y + 7 = 0$

Given $6x + 8y + 7 = 0$

$$m = \frac{\text{co}-\text{efficient of } x}{\text{co}-\text{efficient of } y}$$

$$= \frac{-6}{8}$$

$$m = \frac{-3}{4}$$

Example 5.31

Find the slope of the line which is

(i) Parallel to $3x - 7y = 11$

$$m = \frac{\text{co}-\text{efficient of } x}{\text{co}-\text{efficient of } y}$$

$$= \frac{-3}{-7}$$

$$m = \frac{3}{7}$$

Since parallel lines have same slopes, slope of any line parallel to $3x - 7y = 11$ is $\frac{3}{7}$

(ii) Perpendicular to $2x - 3y + 8 = 0$

$$m = \frac{\text{co}-\text{efficient of } x}{\text{co}-\text{efficient of } y}$$

$$= \frac{-2}{-3}$$

$$m = \frac{2}{3}$$

Slope of a line perpendicular to given line is $\frac{-1}{m}$

$\therefore$ Required $m = -3/2$

---

2. Find the slope of the line which is

(i) Parallel to $y = 0.7x - 11$

(ii) Perpendicular to the line $x = -11$

(i) Given

$$y = 0.7x - 11$$

$$0.7x - y - 11 = 0$$

$$\text{Slope } = \frac{\text{co}-\text{efficient of } x}{\text{co}-\text{efficient of } y}$$

$$= \frac{-0.7}{-1}$$

$$m = 0.7$$

Slope of a line parallel to this line is 0.7.

(ii) Given

$$x = -11$$

$$x + 11 = 0$$

$$\text{Slope } = -\frac{\text{co}-\text{efficient of } x}{\text{co}-\text{efficient of } y}$$

$$= \frac{-1}{0}$$

$$= \infty \text{ (not defined)}$$

slope of a line perpendicular to given line is

$$\frac{-1}{m} = \frac{-1}{-1/0}$$

$$= -1 \times \frac{0}{-1}$$

$$\boxed{m = 0}$$

3. Check whether the given lines are parallel or perpendicular. (i) $\frac{x}{3}+\frac{y}{4}+\frac{1}{7}=0$ and $\frac{2x}{3}+\frac{y}{2}+\frac{1}{10}=0$

Line 1	Line 2
$\frac{x}{3}+\frac{y}{4}+\frac{1}{7}=0$	$\frac{2x}{3}+\frac{y}{2}+\frac{1}{10}=0$
$m = \frac{\text{co}-\text{eff } x}{\text{co}-\text{eff } y}$	$m = \frac{\text{co}-\text{eff } x}{\text{co}-\text{eff } y}$
$= \frac{-1/3}{1/4}$	$= \frac{\frac{-2}{3}}{\frac{1}{2}}$
$= \frac{-1}{3}\times\frac{4}{1}$	$= \frac{-2}{3}\times\frac{2}{1}$
$m_1 = -4/3$	$m_2 = -4/3$

Here $m_1 = m_2$

Hence given lines are parallel.

(ii) $5x + 23y + 14 = 0$ and $23x - 5y + 9 = 0$

Line 1	Line 2
$5x + 23y + 14 = 0$	$23x - 5y + 9 = 0$
$m = \frac{\text{co}-\text{eff } x}{\text{co}-\text{eff } y}$	$m = \frac{\text{co}-\text{eff } x}{\text{co}-\text{eff } y}$
$= \frac{-5}{23}$	$= \frac{-23}{-5}$
	$m_2 = \frac{23}{5}$

Here $m_1 \times m_2 = \frac{-5}{23}\times\frac{23}{5}$

$= -1$

$\therefore$ Given lines are perpendicular.

Example 5.32

Show that the straight lines $2x + 3y - 8 = 0$ and $4x + 6y + 18 = 0$ are parallel.

[Another method]

Given

- $2x + 3y - 8 = 0$

  $a_1 = 2$; $b_1 = 3$; $c_1 = -8$

- $4x + 6y + 18 = 0$

  $a_2 = 4$; $b_2 = 6$; $c_2 = 18$

Now $\frac{a_1}{a_2} = \frac{2}{4} = \frac{1}{2}$

$\frac{b_1}{b_2} = \frac{3}{6} = \frac{1}{2}$

we get $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

Hence given lines are parallel.

Example 5.33

Show that the straight lines $x - 2y + 3 = 0$ and $6x + 3y + 8 = 0$ are perpendicular.

Given:

$x - 2y + 3 = 0$ and $6x + 3y + 8 = 0$

$a_1 = 1$; $b_1 = -2$

$a_2 = 6$; $b_2 = 3$

$\therefore a_1 a_2 + b_1 b_2 = (1)(6) + (-2)(3)$

$= 6 - 6$

$= 0$

$\therefore$ The lines are perpendicular.

4. If the straight lines $12y = (p + 3)x + 12$, $12x - 7y = 16$ are perpendicular the find P.

Given:

- $12y = -(p + 3)x + 12$

$$(p+3)\,x + 12y - 12 = 0$$

$$a_1 = p+3$$

$$b_1 = 12$$

- $12x - 7y = 16$

$$12x - 7y - 16 = 0$$

$$a_2 = 12$$

$$b_2 = -7$$

Given lines are perpendicular

$$\therefore a_1 a_2 + b_1 b_2 = 0$$

$$(P+3)\,12 + (12)(-7) = 0$$

$$12P + 36 - 84 = 0$$

$$12P - 48 = 0$$

$$12P = 48$$

$$P = \frac{48}{12}$$

$$\boxed{P = 4}$$

Type: II Find the equation of a straight line.

Case (i) Parallel, perpendicular form QNo: 5, Example 5.34, 5.35, 6, 7, 8, 9, 10, Example 5.36.

Case (ii) Two points form Q.No.11, 12, Example 5.37.

5. Find the equation of a straight line passing through the point $P(-5, 2)$ and parallel to the line joining the points $Q(3, -2)$ and $R(-5, 4)$

Given line: $Q \begin{matrix} (3, & -2) \\ x_1 & y_1 \end{matrix}$ $R \begin{matrix} (-5, & 4) \\ x_2 & y_2 \end{matrix}$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{4+2}{-5-3}$$

$$= \frac{6}{-8}$$

$$m = \frac{-3}{4}$$

Slope of a line parallel to this line QR is $= \frac{-3}{4}$

$\therefore$ **equation**

$m = -3/4$ point $P(-5 \quad 2)$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{-3}{4}(x+5)$$

$$4(y-2) = -3(x+5)$$

$$4y - 8 = -3x - 15$$

$$4y - 8 + 3x + 15 = 0$$

$$\boxed{\mathbf{3x + 4y + 7 = 0}}$$

Example 5.34

Find the equation of a straight line which is parallel to the line $3x - 7y = 12$ and passing through the point (6,4).

Given line

$$3x - 7y - 12 = 0$$

equation of a line parallel to the given line is of the form

$$3x - 7y + k = 0 \qquad \text{...(1)}$$

(6,4) passes through equ (1)

$$\therefore 3(6) - 7(4) + k = 0$$

$$18 - 28 + k = 0$$

$$-10 + k = 0$$

$$k = 10$$

$\therefore$ (1) $\Rightarrow$ $\mathbf{3x - 7y + 10 = 0}$

Example 5.35

Find the equation of s straight line perpendicular to the line $y = \frac{4}{3}x - 7$ and passing through the point (7, – 1)

Given line

$$y = \frac{4}{3}x - 7$$

$$3y = 4x - 21$$

$$4x - 3y - 21 = 0$$

equation of a line perpendicular to the given line is of the form.

$$3x + 4y + k = 0 \quad \text{...(1)}$$

(7, – 1) passes through equ (1)

$$3(7) + 4(-1) + k = 0$$

$$21 - 4 + k = 0$$

$$17 + k = 0$$

$$k = -17$$

$\therefore$ (1) $\Rightarrow$ $\mathbf{3x + 4y - 17 = 0}$

---

6. Find the equation of a line passing through (6 – 2) and perpendicular to the line joining the points (6,7) and (2, – 3).

Given line joining the points.

$A(6, 7)$ and $B(2, -3)$

$$\text{Slope of } AB = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-3-7}{2-6}$$

$$= \frac{-10}{-4}$$

$$m = \frac{5}{2}$$

$$\text{Perpendicular slope } = \frac{-1}{m}$$

$$= \frac{-2}{5}$$

$\therefore$ **equation**

$$\begin{matrix} (6 & -2) \\ x_1 & y_1 \end{matrix} \quad m = \frac{-2}{5}$$

$$y - y_1 = m(x - x_1)$$

$$y + 2 = \frac{-2}{5}(x - 6)$$

$$5(y + 2) = -2(x - 6)$$

$$5y + 10 = -2x + 12$$

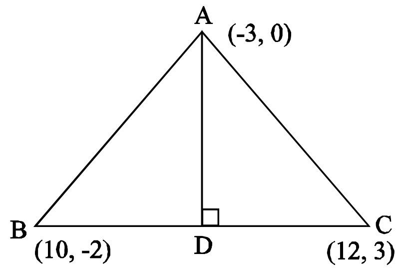
$$5y + 10 + 2x - 12 = 0$$

$$\boxed{\mathbf{2x + 5y - 2 = 0}}$$

---

7. A (– 3, 0), *B* (10, – 2) and *C* (12, 3) are the vertices of ΔABC. Find the equation of the attitude through *A* and *B*.

(i) Equation of attitude through A



$$\text{Slope of } BC = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{3+4}{12-10}$$

$$= \frac{5}{2}$$

slope of attitude AD = perpendicular slope of BC

$$m = \frac{-2}{5}$$

$\therefore$ **Equation of attitude AD**

$$A(-3, 0)\ m = \frac{-2}{5}$$

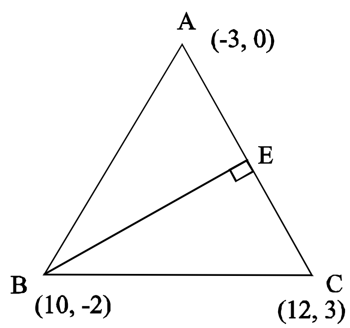
$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{-2}{5}(x+3)$$

$$5y = -2x - 6$$

$$\boxed{\mathbf{2x + 5y + 6 = 0}}$$

(ii) Equation of attitude through *B*



Slope of $AC = \frac{3-0}{12+3}$

$= \frac{3}{15}$

$= \frac{1}{5}$

Slope of attitude $BE =$ perpendicular slope of AC

$m = -5$

Equation of attitude BE

$$B\,(10, -2)\ m = -5$$

$$y - y_1 = m\,(x - x_1)$$

$$y + 2 = -5\,(x - 10)$$

$$y + 2 = -5x + 50$$

$$5x + y + 2 - 50 = 0$$

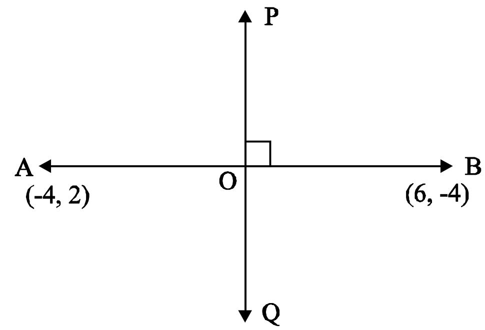
$$\therefore\ \mathbf{5x + y - 48 = 0}$$

---

8. Find the equation of the perpendicular bisector of the line joining the points *A* (– 4, 2) and *B* (6, – 4)

A line which is perpendicular and bisects the given line is known as perpendicular bisector.

$\therefore M =$ midpoint of AB



$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$= \left(\frac{-4+6}{2}, \frac{2-4}{2}\right)$

$= \left(\frac{2}{2}, \frac{-2}{2}\right)$

$= (1, -1)$

Slope of $AB = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-4-2}{6+4}$

$= \frac{-6}{10}$

$= \frac{-3}{5}$

Slope of $PQ =$ perpendicular slope of AB

$= \frac{5}{3}$

$\therefore$ equation of perpendicular bisector

$$(1, -1)\ m = 5/3$$

$$y - y_1 = m\,(x - x_1)$$

$$y + 1 = \frac{5}{3}(x - 1)$$

$$3y + 3 = 5x - 5$$

$$5x + 3y + 3 + 5 = 0$$

$$\mathbf{5x + 3y + 8 = 0}$$

9. Find the equation of a straight line through the intersection of lines $7x + 3y = 10, 5x - 4y = 1$ and parallel to the line $13x + 5y + 12 = 0$

To find point of intersection solve

$$7x + 3y = 10 \quad ...(1)$$

$$5x - 4y = 1 \quad ...(2)$$

$$(1) \times 4 \Rightarrow 28x + 12y = 40$$

$$(2) = 3 \Rightarrow 15x - 12y = 3$$

$$43x = 43$$

$$\boxed{x = 1}$$

put $x = 1$ in (1)

$$7\,(1) + 3y = 10$$

$$7 + 3y = 10$$

$$3y = 10 - 7$$

$$3y = 3$$

$$\boxed{y = 1}$$

$\therefore$ point of intersection (1,1)

Given line

$$13x + 5y + 12 = 0 \quad ...(3)$$

Any line parallel to the given line is of the form

$$13x + 5y + k = 0 \quad ...(4)$$

(1,1) passes through equ (4)

$$13\,(1) + 5\,(1) + k = 0$$

$$13 + 5 + k = 0$$

$$18 + k = 0$$

$$k = -18$$

$\therefore$ Required parallel line is

$$(4) \Rightarrow \mathbf{13x + 5y - 18 = 0}$$

10. Find the equation of a straight line through the intersection of lines $5x - 6y = 2,\ 3x + 2y = 10$ and perpendicular to the line $4x - 7y + 13 = 0$.

To find point of intersection

Solve $5x - 6y = 2 \quad ...(1)$

$$3x + 2y = 10 \quad ...(2)$$

$$(1) \Rightarrow 5x - 6y = 2$$

$$(2) \times \Rightarrow 9x + 6y = 30$$

$$14x = 32$$

$$x = \frac{32}{14}$$

$$\boxed{x = \frac{16}{7}}$$

Put $x = \frac{16}{7}$ in (2)

$$3\left(\frac{16}{7}\right) + 2y = 10$$

$$\frac{48}{7} + 2y = 10$$

$$2y = 10 - \frac{48}{7}$$

$$2y = \frac{22}{7}$$

$$y = \frac{22}{7 \times 2}$$

$$\boxed{y = \frac{11}{7}}$$

$\therefore$ point of intersection $\left(\frac{16}{7}, \frac{11}{7}\right)$

Any line perpendicular to the line $4x - 7y + 13 = 0$ is of the form

$$7x + 4y + k = 0 \quad ...(3)$$

$\left(\frac{16}{7}, \frac{11}{7}\right)$ passes through (3)

$$7\left(\frac{16}{7}\right) + 4\left(\frac{11}{7}\right) + k = 0$$

$$\frac{112}{7} + \frac{44}{7} + k = 0$$

$$\frac{156}{7} + k = 0$$

$$k = \frac{-156}{7}$$

$\therefore$ Required equation

$$(3) \Rightarrow 7x + 4y - \frac{156}{7} = 0$$

$$\boxed{49x + 28y - 156 = 0}$$

Example 5.36

Find the equation of a straight line parallel to y-axis and passing through the point of intersection of the lines $4x + 5y = 13$ and $x - 8y + 9 = 0$

Given liens:

$$4x + 5y - 13 = 0 \quad ...(1)$$

$$x - 8y + 9 = 0 \quad ...(2)$$

To find point of intersection solve (1) & (2)

$$\begin{array}{cccc} x & y & 1 & \\ 5 & -13 & 4 & 5 \\ -8 & 9 & 1 & -8 \end{array}$$

$$\frac{x}{45 - 104} = \frac{y}{-13 - 36} = \frac{1}{-32 - 5}$$

$$\frac{x}{-59} = \frac{y}{-49} = \frac{1}{-37}$$

$$\frac{x}{59} = \frac{1}{37} \qquad \frac{y}{49} = \frac{1}{37}$$

$$x = \frac{59}{37} \qquad y = \frac{49}{37}$$

$\therefore$ pint of intersection $\left(\frac{59}{37}, \frac{49}{37}\right)$

The equation of line parallel to y-axis is $x = c$

It passes through $\left(\frac{59}{37}, \frac{49}{37}\right)$

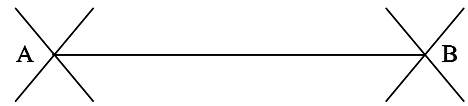
So $C = \frac{59}{37}$

$\therefore$ equation $x = \frac{59}{37}$

$$37x - 59 = 0$$

---

11. Find the equation of a straight line joining the point of intersection of $3x + y + 2 = 0$ and $x - 2y - 4 = 0$ to the point of intersection of $7x - 3y = -12$ and $2y = x + 3$



Solve the equations

$$3x + y = -2 \quad ...(1)$$

$$x - 2y = 4 \quad ...(2)$$

$$(1) \times 2 \Rightarrow 6x + 2y = -4$$

$$(2) \Rightarrow x - 2y = 4$$

$$7x = 0$$

$$\boxed{x = 0}$$

put $x = 0$ in (1)

$$3(0) + y = -2$$

$$\boxed{y = -2}$$

$\therefore$ point of intersection $A(0, -2)$

Solve the equations

$$7x - 3y = -12 \quad ...(3)$$

$$2y = x + 3$$

$$-x + 2y = 3 \quad ...(4)$$

$$(3) \Rightarrow 7x - 3y = -12$$

$$(4) \times 7 \Rightarrow -7x + 14y = 21$$

$$11y = 9$$

$$\boxed{y = \frac{9}{11}}$$

put $y = 9/11$ in (4)

$$-x + 2\left(\frac{9}{11}\right) = 3$$

$$-x + \frac{18}{11} = 3$$

$$\frac{18}{11} - 3 = x$$

$$\boxed{\mathbf{\frac{-15}{11} = x}}$$

$\therefore$ point of intersection $B\left(\frac{-15}{11}, \frac{9}{11}\right)$

$\therefore$ Equation $A(0, -2)\ B\left(\frac{-15}{11}, \frac{9}{11}\right)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y + 2}{\frac{9}{11} + 2} = \frac{x - 0}{\frac{-15}{11} - 0}$$

$$\frac{y + 2}{\frac{31}{11}} = \frac{x}{\frac{-15}{11}}$$

$$31x = -15(y + 2)$$

$$31x = -15y - 30$$

$$\boxed{\mathbf{31x + 15y + 30 = 0}}$$

12. Find the equation of a straight line through the point of intersection of the lines s$8x + 3y = 18$, $4x + 5y = 9$ and bisecting the line segment joining the points $(5, -4)$ and $(-7, 6)$

Given lines

$$8x + 3y - 18 = 0 \quad \text{...(1)}$$

$$4x + 5y - 9 = 0 \quad \text{...(2)}$$

solve (1), (2)

$$\begin{array}{cccccc} & x & & y & & 1 \\ 3 & & -18 & & 8 & 3 \\ 5 & & -9 & & 4 & 5 \end{array}$$

$$\frac{x}{-27 + 90} = \frac{y}{-72 + 72} = \frac{1}{40 - 12}$$

$$\frac{x}{63} = \frac{y}{0} = \frac{1}{28}$$

$$\frac{x}{63} = \frac{1}{28} \qquad \frac{y}{0} = \frac{1}{28}$$

$$\frac{x}{9} = \frac{1}{4} \qquad 28y = 0$$

$$\mathbf{y = 0}$$

$$\mathbf{x = \frac{9}{4}}$$

$\therefore$ point of intersection $\left(\frac{9}{4}, 0\right)$

- Midpoint of $(5, -4)$ and $(-7, 6)$

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{5 - 7}{2}, \frac{-4 + 6}{2}\right)$$

$$= \left(\frac{-2}{2}, \frac{2}{2}\right)$$

$$= (-1, 1)$$

$\therefore$ **equation**

$$\begin{pmatrix} \frac{9}{4} & , 0) \\ x_1 & y_1 \end{pmatrix} \text{ and } \begin{array}{cc} (-1 & 1) \\ x_2 & y_2 \end{array}$$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 0}{1 - 0} = \frac{x - 9/4}{-1 - 9/4}$$

$$\frac{y}{1} = \frac{\frac{4x - 9}{4}}{\frac{-4 - 9}{4}}$$

$$\frac{y}{1} = \frac{4x - 9}{-13}$$

$$4x - 9 = -13y$$

$$\boxed{\mathbf{4x + 13y - 9 = 0}}$$

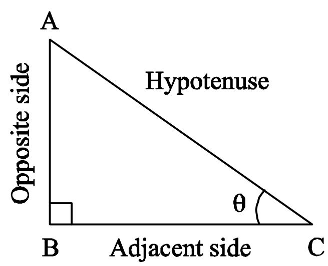
CHAPTER 6

TRIGONOMETRY

Exercise 6.1

KEY POINTS

I. Trigonometric ratios



1. $\sin\theta = \dfrac{\text{Opposite side}}{\text{Hypotenuse}}$

2. $\cos\theta = \dfrac{\text{Adjacent side}}{\text{Hypotenuse}}$

3. $\tan\theta = \dfrac{\text{Opposite side}}{\text{Adjacent side}}$

4. $\text{cosec}\,\theta = \dfrac{\text{Hypotenuse}}{\text{Opposite side}}$

5. $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Opposite side}}$

6. $\cot\theta = \dfrac{\text{Adjacent side}}{\text{Opposite side}}$

II. Reciprocal relation

1. $\sin\theta = \dfrac{1}{\text{cosec}\,\theta}$; $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$

2. $\cos\theta = \dfrac{1}{\sec\theta}$; $\sec\theta = \dfrac{1}{\cos\theta}$

3. $\tan\theta = \dfrac{1}{\cot\theta}$; $\cot\theta = \dfrac{1}{\tan\theta}$

III. Quotient relation

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

2. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Note

- $\sin\theta\,\text{cosec}\,\theta = 1$
- $\cos\theta\sec\theta = 1$
- $\tan\theta\cot\theta = 1$

IV. Complementary angles

1. $\sin(90-\theta) = \cos\theta$; $\cos(90-\theta) = \sin\theta$

2. $\sec(90-\theta) = \text{cosec}\,\theta$; $\text{cosec}\,(90-\theta) = \sec\theta$

3. $\tan(90-\theta) = \cot\theta$; $\cot(90-\theta) = \tan\theta$

V. Trigonometric table

θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cosec θ	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec θ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cot θ	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

VI. Trigonometric identities

1. $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\cos^2\theta = 1 - \sin^2\theta$$

2. $1 + \tan^2\theta = \sec^2\theta$

$$\tan^2\theta = \sec^2\theta - 1$$

$$1 = \sec^2\theta - \tan^2\theta$$

3. $1 + \cot^2\theta = \mathrm{cosec}^2\theta$

$$\cot^2\theta = \mathrm{cosec}^2\theta - 1$$

$$1 = \mathrm{cosec}^2\theta - \cot^2\theta$$

Note

Algebraic identities

(i) $a^2 - b^2 = (a+b)(a-b)$

(ii) $a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$

(iii) $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$

Type I: Trigonometric identities, Reciprocal relation based sums

Q.No: 5.(i) (ii), 2.(i) (ii), Example 6.1, 6.4, 6.6, 6.9, 5.(i) (ii), Example 6.15, 6.16, 6.3

1. Prove the following identities.

(i) $\cot\theta + \tan\theta = \sec\theta\,\mathrm{cosec}\,\theta$

LHS

$$\cot\theta + \tan\theta$$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$= \sec\theta\,\mathrm{cosec}\,\theta$ **RHS**

(ii) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

LHS

$$\tan^4\theta + \tan^2\theta$$

$$= \tan^2\theta(\tan^2\theta + 1)$$

$$= \tan^2\theta \cdot \sec^2\theta$$

RHS

$$\sec^4\theta - \sec^2\theta$$

$$= \sec^2\theta(\sec^2\theta - 1)$$

$$= \sec^2\theta \cdot \tan^2\theta$$

$$\text{LHS} = \text{RHS}$$

Hence proved

2. Prove the following identities.

(i) $\dfrac{1 - \tan^2\theta}{\cot^2\theta - 1} = \tan^2\theta$

LHS

$$\frac{1 - \tan^2\theta}{\cot^2\theta - 1}$$

$$= \frac{1 - (\sec^2\theta - 1)}{(\mathrm{cosec}^2\theta - 1) - 1}$$

$$= \frac{1 - \sec^2\theta + 1}{\mathrm{cosec}^2\theta - 1 - 1}$$

$$= \frac{2 - \sec^2\theta}{\mathrm{cosec}^2\theta - 2}$$

$$= \frac{2 - \dfrac{1}{\cos^2\theta}}{\dfrac{1}{\sin^2\theta} - 2}$$

$$= \frac{\dfrac{2\cos^2\theta - 1}{\cos^2\theta}}{\dfrac{1 - 2\sin^2\theta}{\sin^2\theta}}$$

$$= \frac{2\cos^2\theta - 1}{\cos^2\theta} \times \frac{\sin^2\theta}{1 - 2\sin^2\theta}$$

$$= \frac{2(1 - \sin^2\theta) - 1}{\cos^2\theta} \times \frac{\sin^2\theta}{1 - 2\sin^2\theta}$$

$$= \frac{2 - 2\sin^2\theta - 1}{\cos^2\theta} \times \frac{\sin^2\theta}{1 - 2\sin^2\theta}$$

$$= \frac{1 - 2\sin^2\theta}{\cos^2\theta} \times \frac{\sin^2\theta}{1 - 2\sin^2\theta}$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$= \tan^2 \theta \quad \text{RHS}$$

(ii) $\mathbf{\dfrac{\cos \theta}{1 + \sin \theta} = \sec \theta - \tan \theta}$

LHS

$$\frac{\cos \theta}{1 + \sin \theta}$$

$$= \frac{\cos \theta}{1 + \sin \theta} \times \frac{1 - \sin \theta}{1 - \sin \theta}$$

$$= \frac{\cos \theta (1 - \sin \theta)}{1 - \sin^2 \theta}$$

$$= \frac{\cos \theta (1 - \sin \theta)}{\cos^2 \theta}$$

$$= \frac{1 - \sin \theta}{\cos \theta}$$

$$= \frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta}$$

$$= \sec \theta - \tan \theta \quad \text{RHS}$$

Example 6.1

Prove that $\mathbf{\tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta}$

LHS

$$\tan^2 \theta - \sin^2 \theta$$

$$= \frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta$$

$$= \frac{\sin^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta}$$

$$= \frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta}$$

$$= \frac{\sin^2 \theta \cdot \sin^2 \theta}{\cos^2 \theta}$$

$$= \tan^2 \theta \cdot \sin^2 \theta \quad \text{RHS}$$

Example 6.3

Prove that $\mathbf{1 + \dfrac{\cot^2 \theta}{1 + \operatorname{cosec} \theta} = \operatorname{cosec} \theta}$

LHS

$$= 1 + \frac{\cot^2 \theta}{1 + \operatorname{cosec} \theta}$$

$$= 1 + \frac{\operatorname{cosec}^2 \theta - 1}{1 + \operatorname{cosec} \theta}$$

$$= \frac{(\operatorname{cosec} \theta + 1)(\operatorname{cosec} \theta - 1)}{1 + \operatorname{cosec} \theta}$$

$$= 1 + \operatorname{cosec} \theta - 1$$

$$= \operatorname{cosec} \theta \quad \text{RHS}$$

Example 6.4

Prove that $\mathbf{\sec \theta - \cos \theta = \tan \theta \sin \theta}$

LHS

$$\sec \theta - \cos \theta$$

$$= \frac{1}{\cos \theta} - \cos \theta$$

$$= \frac{1 - \cos^2 \theta}{\cos \theta}$$

$$= \frac{\sin^2 \theta}{\cos \theta}$$

$$= \frac{\sin \theta \cdot \sin \theta}{\cos \theta}$$

$$= \tan \theta \cdot \sin \theta \quad \textbf{RHS}$$

Example 6.6

Prove that $\mathbf{\dfrac{\sec \theta}{\sin \theta} - \dfrac{\sin \theta}{\cos \theta} = \cot \theta}$

LHS

$$\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta}$$

$$= \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1-\sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$= \cot\theta$ RHS

Example 6.9

Prove that

$(\csc\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

LHS

$(\csc\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \left(\frac{1-\sin^2\theta}{\sin\theta}\right)\left(\frac{1-\cos^2\theta}{\cos\theta}\right)\left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)$$

$$= \frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^2\theta}{\cos\theta} \times \frac{1}{\cos\theta\sin\theta}$$

$$= \frac{\cos^2\theta\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$= 1$ RHS

5. Prove that following identities

(i) $\sec^4\theta(1-\sin^4\theta) - 2\tan^2\theta = 1$

LHS

$$\sec^4\theta(1-\sin^4\theta) - 2\tan^2\theta$$

$$= \sec^4\theta(1-\sin^2\theta)(1+\sin^2\theta) - 2\tan^2\theta$$

$$= \sec^4\theta(\cos^2\theta)(1+\sin^2\theta) - 2\tan^2\theta$$

$$= \frac{1}{\cos^4\theta}(\cos^2\theta)(1+\sin^2\theta) - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1+\sin^2\theta}{\cos^2\theta} - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1+\sin^2\theta - 2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1-\sin^2\theta}{\cos^2\theta}$$

$$= \frac{\cos^2\theta}{\cos^2\theta}$$

$= 1$ RHS

(ii) $\dfrac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \dfrac{\csc\theta - 1}{\csc\theta + 1}$

LHS

$$\frac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta}$$

$$= \frac{\frac{\cos\theta}{\sin\theta} - \cos\theta}{\frac{\cos\theta}{\sin\theta} + \cos\theta}$$

$$= \frac{\frac{\cos\theta - \cos\theta\sin\theta}{\sin\theta}}{\frac{\cos\theta + \cos\theta\sin\theta}{\sin\theta}}$$

$$= \frac{\cos\theta(1-\sin\theta)}{\cos\theta(1+\sin\theta)}$$

$$= \frac{1-\sin\theta}{1+\sin\theta}$$

$$= \frac{1 - \frac{1}{\csc\theta}}{1 + \frac{1}{\csc\theta}}$$

$$= \frac{\frac{\csc\theta - 1}{\csc\theta}}{\frac{\csc\theta + 1}{\csc\theta}}$$

$= \dfrac{\csc\theta - 1}{\csc\theta + 1}$ RHS

Example 6.15

Show that $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1-\tan A}{1-\cot A}\right)^2$

LHS

$$\frac{1+\tan^2 A}{1+\cot^2 A}$$

$$= \frac{1+\tan^2 A}{1+\dfrac{1}{\tan^2 A}}$$

$$= \frac{1+\tan^2 A}{\dfrac{\tan^2 A+1}{\tan^2 A}}$$

$$= \tan^2 A$$

RHS

$$\left(\frac{1-\tan A}{1-\cot A}\right)^2$$

$$= \left(\frac{1-\tan A}{1-\dfrac{1}{\tan A}}\right)^2$$

$$= \left(\frac{1-\tan A}{\dfrac{\tan A-1}{\tan A}}\right)^2$$

$$= (-\tan A)^2$$

$$= \tan^2 A$$

$$\text{LHS} = \text{RHS}$$

Hence proved.

Example 6.16

Prove that

$$\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}$$

LHS

$$\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}$$

$$= \frac{\left(1+\dfrac{\cos A}{\sin A}+\dfrac{\sin A}{\cos A}\right)(\sin A-\cos A)}{\dfrac{1}{\cos^3 A}-\dfrac{1}{\sin^3 A}}$$

$$= \frac{\dfrac{(\sin A\cos A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cos A}}{\dfrac{\sin^3 A-\cos^3 A}{\cos^3 A\sin^3 A}}$$

$$= \frac{(\sin A\cos A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cos A} \times \frac{\cos^3 A\sin^3 A}{\sin^3 A-\cos^3 A}$$

we have $(a-b)(a^2+ab+b^2) = a^3-b^3$

$$= \frac{\sin^3 A-\cos^3 A}{\sin A\cos A} \times \frac{\cos^3 A\sin^3 A}{\sin^3 A-\cos^3 A}$$

$$= \sin^2 A\cos^2 A \quad \text{RHS}$$

Type I: Conjugate multiplication based sums.

Q.No. 3, (i) (ii), Example 6.2, 6.3, 6.5

3. Prove the following identities

(i) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sec\theta+\tan\theta$

LHS

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}}$$

$$\sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1+\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta \quad \text{RHS}$$

(ii) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$

LHS

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} + \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1+\sin^2\theta}{\cos^2\theta}} + \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1+\sin\theta}{\cos\theta} + \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1+\sin\theta+1-\sin\theta}{\cos\theta}$$

$$= \frac{2}{\cos\theta}$$

$$= 2\sec\theta \quad \text{RHS}$$

Example 6.2

Prove that $\dfrac{\sin A}{1+\cos A} = \dfrac{1-\cot A}{\sin A}$

LHS

$$\frac{\sin A}{1+\cos A}$$

$$= \frac{\sin A}{1+\cos A} \times \frac{1-\cos A}{1-\cos A}$$

$$= \frac{\sin A\,(1-\cos A)}{1-\cos^2 A)}$$

$$= \frac{\sin A\,(1-\cos A)}{\sin^2 A}$$

$$= \frac{1-\cos A}{\sin A} \quad \text{RHS}$$

Example 6.3

Prove that $1 + \dfrac{\cot^2\theta}{1+\operatorname{cosec}\theta} = \operatorname{cosec}\theta$

LHS

$$= 1 + \frac{\cot^2\theta}{1+\operatorname{cosec}\theta}$$

$$= 1 + \frac{\operatorname{cosec}^2\theta - 1}{1+\operatorname{cosec}\theta}$$

$$= 1 + \frac{(\operatorname{cosec}\theta+1)(\operatorname{cosec}\theta-1)}{1+\operatorname{cosec}\theta}$$

$$= 1 + \operatorname{cosec}\theta - 1$$

$$= \operatorname{cosec}\theta \quad \text{RHS}$$

Example 6.5

Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

LHS

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$= \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$

$$= \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$= \operatorname{cosec}\theta + \cot\theta$ RHS

Type III: Cross multiplication based sums (LCM)

Q.No: 6, (i) (ii) Example 6.10, 6.12, 6.14

6. Prove the following identities

(i) $$\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$$

LHS

$$\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$= \frac{(\sin A - \sin B)(\sin A + \sin B) + (\cos A - \cos B)(\cos A + \cos B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{\sin^2 A - \sin^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{(1-\cos^2 A) - (1-\cos^2 B) + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{(1-\cos^2 A) - (1-\cos^2 B) + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{1-\cos^2 A - 1 + \cos^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$= 0$ RHS

(ii) $$\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$$

LHS

$$\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A}$$

$$= \frac{(\sin A + \cos A)(\sin^2 A + \cos^2 A - \sin A\cos A)}{(\sin A + \cos A)} + \frac{(\sin A - \cos A)(\sin^2 A + \cos^2 A + \sin A\cos A)}{\sin A - \cos A}$$

$= 1 - \sin A\cos A + 1 + \sin A\cos A$

$= 1 + 1$

$= 2$ RHS

Example 6.10

Prove that $$\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A} = 2\operatorname{cosec} A$$

LHS

$$\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A}$$

$$= \frac{\sin A(1-\cos A) + \sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$$

$$= \frac{\sin A - \sin A\cos A + \sin A + \sin A\cos A}{1-\cos^2 A}$$

$$= \frac{2\sin A}{\sin^2 A}$$

$$= \frac{2}{\sin A}$$

$= 2\operatorname{cosec} A$ RHS

Example 6.12

Prove that $$\tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$$

LHS

$\tan^2 A - \tan^2 B$

$$= \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$= \frac{\sin^2 A\cos^2 B - \cos^2 A\sin^2 B}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B}{\cos^2 A \cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B} \quad \text{RHS}$$

Example 6.14

$$\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\operatorname{cosec} A + \cot A - 1} = 1$$

LHS

$$\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\operatorname{cosec} A + \cot A - 1}$$

$$= \frac{\sin A}{\frac{1}{\cos A} + \frac{\sin A}{\cos A} - 1} + \frac{\cos A}{\frac{1}{\sin A} + \frac{\cos A}{\sin A} - 1}$$

$$= \frac{\sin A}{\frac{1 + \sin A - \cos A}{\cos A}} + \frac{\cos A}{\frac{1 + \cos A - \sin A}{\sin A}}$$

$$= \frac{\sin A \cos A}{1 + \sin A - \cos A} + \frac{\sin A \cos A}{1 + \cos A - \sin A}$$

$$= \frac{\sin A \cos A\,(1 + \cos A - \sin A) + \sin A \cos A\,(1 + \sin A - \cos A)}{(1 + \sin A - \cos A)\,(1 + \cos A - \sin A)}$$

$$= \frac{\sin A \cos A + \sin A \cos^2 A - \sin^2 A \cos A + \sin A \cos A + \sin^2 A \cos A - \sin A \cos^2 A}{[1 + (\sin A - \cos A)]\,[1 - [\sin A - \cos A]]}$$

$$= \frac{2 \sin A \cos A}{1^2 - (\sin A - \cos A)^2}$$

$$= \frac{2 \sin A \cos A}{1 - (\sin^2 A + \cos^2 A - 2 \sin A \cos A)}$$

$$= \frac{2 \sin A \cos A}{1 - (1 - 2 \sin A \cos A)}$$

$$= \frac{2 \sin A \cos A}{1 - 1 + 2 \sin A \cos A}$$

$$= \frac{2 \sin A \cos A}{2 \sin A \cos A}$$

$= 1$ RHS

Type IV: Algebra identities based sums
Q.No: 4, (i) (ii), Example 6.13

4. Prove the following identities:

(i) $\sec^6 \theta = \tan^6 \theta + 3 \tan^2 \theta \sec^2 \theta + 1$

LHS

$\sec^6 \theta$

$$= (\sec^2 \theta)^3$$

$$= (1 + \tan^2 \theta)^3$$

use $(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$

$$= 1 + 3 \tan^2 \theta + 3 \tan^4 \theta + \tan^6 \theta$$

$$= 1 + 3 \tan^2 \theta\,(1 + \tan^2 \theta) + \tan^6 \theta$$

$$= 1 + 3 \tan^2 \theta \cdot \sec^2 \theta + \tan^6 \theta \quad \text{RHS}$$

(ii) $(\sin \theta + \sec \theta)^2 + (\cos \theta + \operatorname{cosec} \theta)^2 = 1 + (\sec \theta + (\operatorname{cosec})^2$

LHS

$(\sin \theta + \sec \theta)^2 + (\cos \theta + \operatorname{cosec} \theta)^2$

$$= \sin^2 \theta + 2 \sin \theta \sec \theta + \sec^2 \theta + \cos^2 \theta + 2 \cos \theta \operatorname{cosec} + \operatorname{cosec}^2 \theta$$

$$= (\sin^2 \theta + \cos^2 \theta) + \sec^2 \theta + \operatorname{cosec}^2 \theta + 2\,[\sin \theta \sec + \cos \theta \operatorname{cosec} \theta]$$

$$= 1 + \sec^2 \theta + \operatorname{cosec}^2 \theta + 2\left(\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}\right)$$

$$= 1 + \sec^2 \theta + \operatorname{cosec}^2 \theta + 2\left(\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta}\right)$$

$$= 1 + \sec^2 \theta + \operatorname{cosec}^2 \theta + 2\left(\frac{1}{\cos \theta \sin \theta}\right)$$

$$= 1 + \sec^2 \theta + \operatorname{cosec}^2 \theta + 2 \sec \theta \operatorname{cosec} \theta$$

$$= 1 + (\sec \theta + \operatorname{cosec} \theta)^2 \quad \text{RHS}$$

Example 6.13

Prove that

$$\left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right) = 2 \sin A \cos A$$

LHS

$$= \left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right)$$

use $a^3 - b^3 = (a + b^3\ (a - b)\ (a^2 + b^2 + ab)$

$a^3 + b^3 = (a + b)\ (a^2 + b^2 - ab)$

$$= \frac{(\cos A - \sin A)\ (\cos^2 A + \sin^2 A + \cos A \sin A)}{(\cos A - \sin A)}$$

$$- \frac{(\cos A + \sin A)\ (\cos^2 A + \sin^2 A - \cos A \sin A)}{\cos A + \sin A}$$

$= \cos^2 A + \sin^2 A + \cos A \sin A - \cos^2 A - \sin^2 A$
$+ \cos A \sin A$

$= 2 \sin A \cos A$ RHS

Type V: Conditional identities:

Q.No: 7, (i) (ii), 8, (i) (ii), 9, (i) (ii), 10
Example 6.8, 6.11, 6.17

7. (i) If $\sin\theta + \cos\theta = \sqrt{3}$, prove that $\tan\theta + \cot\theta = 1$.

Given $\sin\theta + \cos\theta = \sqrt{3}$

Squaring on both sides

$$(\sin\theta + \cos\theta)^2 = (\sqrt{3})^2$$

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 3$$

$$1 + 2\sin\theta\cos\theta = 3$$

$$2\sin\theta\cos\theta = 3 - 1$$

$$2\sin\theta\cos\theta = 2$$

$$\sin\theta\cos\theta = \frac{2}{2}$$

$$\sin\theta\cos\theta = 1 \qquad ...(1)$$

Let **LHS**

$$\tan\theta + \cot\theta$$

$$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \qquad \text{(From (1))}$$

$$= \frac{1}{1}$$

$= 1$ RHS

Hence proved.

7. (ii) If $\sqrt{3}\sin\theta - \cos\theta = 0$ then show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Given $\sqrt{3}\sin\theta - \cos\theta = 0$

$$\sqrt{3}\sin\theta = \cos\theta$$

$$\sqrt{3} = \frac{\cos\theta}{\sin\theta}$$

$$\sqrt{3} = \cot\theta$$

$$\boxed{\theta = 30°}$$

LHS

$$\tan 3\theta = \tan 3(30°)$$

$$= \tan 90°$$

$$= \infty \text{ (not defined)} \qquad ...(1)$$

RHS

$$\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \frac{3\tan 30° - \tan^3 30°}{1 - 3\tan^2 30°}$$

$$= \frac{3\left(\frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{3}}\right)^3}{1 - 3\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}}}{1 - 3\left(\frac{1}{3}\right)}$$

$$= \frac{\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}}}{1 - 1}$$

$$= \frac{\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}}}{0}$$

$= \infty$ (not defined) ...(2)

From (1) and (2) proved.

8. (i) If $\frac{\cos\alpha}{\cos\beta} = m$ and $\frac{\cos\alpha}{\sin\beta} = n$ then prove that $(m^2 + n^2)\cos^2\beta = n^2$

Given

$$m = \frac{\cos\alpha}{\cos\beta} \qquad n = \frac{\cos\alpha}{\sin\beta}$$

$$m^2 = \frac{\cos^2\alpha}{\cos^2\beta} \qquad n^2 = \frac{\cos^2\alpha}{\sin^2\beta}$$

LHS

$$(m^2 + n^2)\cos^2\beta$$

$$= \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta$$

$$= \left(\frac{\cos^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta}{\cos^2\beta\sin^2\beta}\right)\cos^2\beta$$

$$= \frac{\cos^2\alpha(\sin^2\beta + \cos^2\beta)}{\sin^2\beta}$$

$$= \frac{\cos^2\alpha}{\sin^2\beta}$$

$= n^2$ RHS

8. (ii) If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$ then prove that $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$

Given

$$x = \cot\theta + \tan\theta$$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$x = \frac{1}{\sin\theta\cos\theta}$$

$$y = \sec\theta - \cos\theta$$

$$= \frac{1}{\cos\theta} - \cos\theta$$

$$= \frac{1 - \cos^2\theta}{\cos\theta}$$

$$= \frac{\sin^2\theta}{\cos\theta}$$

LHS

$$(x^2y)^{2/3} - (xy^2)^{2/3}$$

$$= \left(\frac{1}{\sin^2\theta\cos^2\theta} \times \frac{\sin^2\theta}{\cos\theta}\right)^{2/3} - \left(\frac{1}{\sin\theta\cos\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right)^{2/3}$$

$$= \left(\frac{1}{\cos^3\theta}\right)^{2/3} - \left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{2/3}$$

$$= (\sec^3\theta)^{2/3} - (\tan^2\theta)^{2/3}$$

$$= \sec^2\theta - \tan^2\theta$$

$= 1$ RHS

9. (i) If $\sin\theta + \cos\theta = p$ and $\sec\theta + \operatorname{cosec}\theta = q$ then prove that $q(p^2 - 1) = 2p$.

Given

$$p = \sin\theta + \cos\theta$$

$$p^2 = (\sin\theta + \cos\theta)^2$$
$$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$$
$$p^2 = 1 + 2\sin\theta\cos\theta$$
$$q = \sec\theta + \operatorname{cosec}\theta = \frac{1}{\cos\theta} + \frac{1}{\sin\theta}$$
$$q = \frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}$$

LHS $q(p^2 - 1)$

$$= \frac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}(1 + 2\sin\theta\cos\theta - 1)$$
$$= \frac{(\sin\theta + \cos\theta)(2\sin\theta\cos\theta)}{\cos\theta\sin\theta)}$$
$$= 2(\sin\theta + \cos\theta) = 2p \quad \text{RHS}$$

---

9. (ii) If $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$, then prove that $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

Given: $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$

$$\sin\theta(1 + 1 - \cos^2\theta) = \cos^2\theta$$
$$\sin\theta(2 - \cos^2\theta = \cos^2\theta$$

Squaring on both sides.

$$\sin^2\theta(2 - \cos^2\theta)^2 = (\cos^2\theta)^2$$
$$(1 + \cos^2\theta)(4 - 4\cos^2\theta + + \cos^4\theta) = \cos^4\theta$$
$$4 - 4\cos^2\theta + \cos^4\theta - 4\cos^2\theta + 4\cos^4\theta - \cos^2\theta\cos^4\theta = \cos^4\theta$$
$$-\cos^6\theta + 4\cos^4\theta - 8\cos^2\theta + 4 = 0$$
$$\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4 \text{ Proved.}$$

---

10. If $\dfrac{\cos\theta}{1 + \sin\theta} = \dfrac{1}{a}$ then prove that $\dfrac{a^2 - 1}{a^2 + 1} = \sin\theta$.

Given: $\dfrac{\cos\theta}{1 + \sin\theta} = \dfrac{1}{a}$

$$a\cos\theta = 1 + \sin\theta$$
$$a = \frac{1 + \sin\theta}{\cos\theta}$$
$$a = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$
$$a = \sec\theta + \tan\theta$$

LHS

$$\frac{a^2 - 1}{a^2 + 1} = \frac{(\sec\theta + \tan\theta)^2 - 1}{(\sec\theta + \tan\theta)^2 + 1}$$
$$= \frac{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta - 1}{\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 1}$$
$$= \frac{\tan^2\theta + \tan^2\theta + 2\sec\theta\tan\theta}{\sec^2\theta + \sec^2\theta + 2\sec\theta\tan\theta}$$
$$= \frac{2\tan^2\theta + 2\sec\theta\tan\theta}{2\sec^2\theta + 2\sec\theta\tan\theta}$$
$$= \frac{2\tan\theta(\tan\theta + \sec\theta)}{2\sec\theta(\sec\theta + \tan\theta)}$$
$$= \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$
$$= \sin\theta \quad \text{RHS}$$

Example 6.8

***If* $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, *then prove that* $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.**

Given: $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

Squaring on both sides

$$(\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$
$$\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta = 2\cos^2\theta$$
$$2\cos\theta\sin\theta = 2\cos^2\theta - \cos^2\theta - \sin^2\theta$$
$$2\cos\theta\sin\theta = \cos^2\theta - \sin^2\theta$$
$$2\cos\theta\sin\theta = (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$$
$$\frac{2\cos\theta\sin\theta}{\cos\theta + \sin\theta} = \cos\theta - \sin\theta$$

Given: $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

$$\frac{2\cos\theta\sin\theta}{\sqrt{2}\cos\theta} = \cos\theta - \sin\theta$$

$$\frac{\sqrt{2}\cdot\sqrt{2}\cdot\cos\theta\sin\theta}{\sqrt{2}\cos\theta} = \cos\theta - \sin\theta$$

$$\sqrt{2}\sin\theta = \cos\theta - \sin\theta$$

Hence proved.

Example 6.11

If $\operatorname{cosec}\theta + \cot\theta = p$, then prove that $\cos\theta = \frac{p^2-1}{p^2+1}$

Given

$$\operatorname{cosec} + \cot\theta = p \quad \text{...(1)}$$

we have

$$\operatorname{cosec}^2\theta - \cot^2\theta = 1$$

$$(\operatorname{cosec}\theta - \cot^2\theta = 1$$

$$(\operatorname{cosec}\theta + \cot\theta)(\operatorname{cosec}\theta - \cot\theta) = 1$$

$$p(\operatorname{cosec}\theta - \cot\theta = 1$$

$$\operatorname{cosec}\theta - \cot\theta = \frac{1}{p} \quad \text{...(2)}$$

Add (1) and (2)

$$\operatorname{cosec} + \cot\theta = p$$

$$\operatorname{cosec}\theta - \cot\theta = \frac{1}{p}$$

$$2\operatorname{cosec}\theta = p + \frac{1}{p}$$

$$2\operatorname{cosec}\theta = \frac{p^2+1}{p} \quad \text{...(3)}$$

Sub (1) and (2)

$$\operatorname{cosec}\theta + \cot\theta = p$$

$$(-) \quad (+) \quad (-)$$

$$\operatorname{cosec}\theta - \cot\theta = \frac{1}{p}$$

$$2\cot\theta = p - \frac{1}{p}$$

$$2\cot\theta = \frac{p^2-1}{p} \quad \text{...(4)}$$

(4) ÷ (3)

$$\frac{2\cot\theta}{2\operatorname{cosec}\theta} = \frac{p^2-1}{p} \times \frac{p}{p^2+1}$$

$$\frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \frac{p^2-1}{p^2+1}$$

$$\cos\theta = \frac{p^2-1}{p^2+1} \text{ proved}$$

Example 6.17

If $\frac{\cos\theta}{\sin\theta} = p$ and $\frac{\sin^2\theta}{\cos\theta} = q$, then prove that $p^2q^2(p^2+q^2+3) = 1$

LHS

$$p^2q^2(p^2+q^2+3)$$

$$= \left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2\left[\left(\frac{\cos^2\theta}{\sin\theta}\right)^2 + \left(\frac{\sin^2\theta}{\cos\theta}\right)^2 + 3\right]$$

$$= \left[\frac{\cos^4\theta}{\sin^2\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right]\left[\frac{\cos^4\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} + 3\right]$$

$$= (\cos^2\theta\cdot\sin^2\theta)\left[\frac{\cos^6\theta + \sin^6\theta + 3\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\right]$$

$$= \cos^6\theta + \sin^6\theta + 3\sin^2\theta\cos^2\theta$$

$$= (\cos^2\theta)^3 + (\sin^2\theta)^3 + 3\sin^2\theta\cos^2\theta$$

use $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$$= [(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)] + 3\sin^2\theta\cos^2\theta$$

$$= (1)^3 - 3\cos^2\theta\sin^2\theta(1) + 3\sin^2\theta\cos^2\theta$$

$$= 1 - 3\cos^2\theta\sin^2\theta + 3\sin^2\theta\cos^2\theta$$

$$= 1 \quad RHS$$

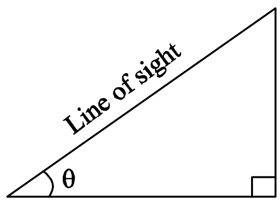
Exercise 6.2

KEY POINTS

Heights and Distances

1. Line of sight

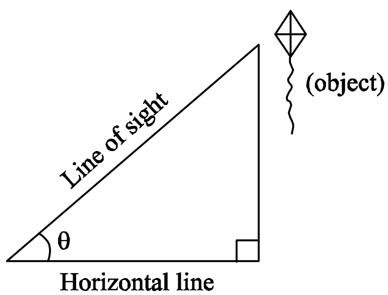
The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.



2. Angle of Elevation

The angle of elevation is angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level.

In this case we raise our head to look at the object.



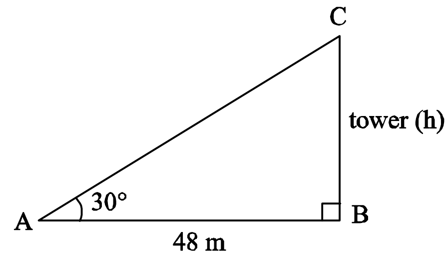
Here θ is the **angle of elevation**.

Type I: Angle of elevation based sums (Given one angle)

Example 6.19, 6.20, 2, 1, Example 6.18

Example 6.19

A tower stands vertically on the ground. From a point on the ground, which is 48 m away from the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower.



✍ **Solution:**

$BC =$ 'h' m height of the tower

$AB = 48$ m Distance from foot of the tower

$$\angle BAC = 30°$$

In ΔABC

$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

$$\tan 30° = \frac{h}{48}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{48}$$

$$\sqrt{3}h = 48$$

$$h = \frac{48}{\sqrt{3}}$$

$$h = \frac{48}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{48\sqrt{3}}{3}$$

$$h = 16\sqrt{3} \text{ m}$$

$\therefore$ Height of the tower is $16\sqrt{3}$ m.

Example 6.20

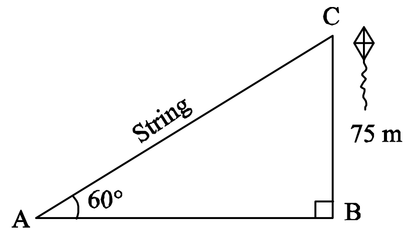
A kite is flying at a height of 75 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.

✍ **Solution:**

$BC = 75$ m height of the kite from the ground

$AC =$ length of string

$\angle BAC = 60°$



In ΔABC

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\sin 60° = \frac{75}{AC}$$

$$\frac{\sqrt{3}}{2} = \frac{75}{AC}$$

$$\sqrt{3}\,AC = 75 \times 2$$

$$AC = \frac{150}{\sqrt{3}}$$

$$= \frac{150}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

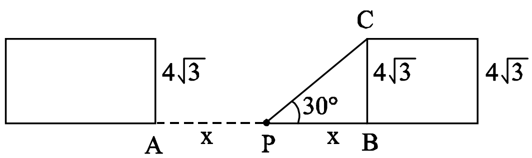
$$= \frac{150\sqrt{3}}{3}$$

$$AC = 50\sqrt{3}\text{ m}$$

$\therefore$ Length of string is $50\sqrt{3}$ m.

2. A road is flanked on either side by continuous rows of houses of height $4\sqrt{3}$ m with no space in between them. A pedestrian is standing on the median of the rod facing a row house. The angle of elevation from the pedestrian to the top of the house is 30°. Find the width of the road.

✍ **Solution:**



$AB =$ width of the rod

$AP = PB = x$ m (mid point of AB is p)

$BC = 4\sqrt{3}$ m height of the house

$$\angle BPC = 30°$$

$$\tan\theta = \frac{\text{Opposite side}}{\text{Adjacent side}}$$

$$\tan 30° = \frac{4\sqrt{3}}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{x}$$

$$x = 4\sqrt{3} \times \sqrt{3}$$

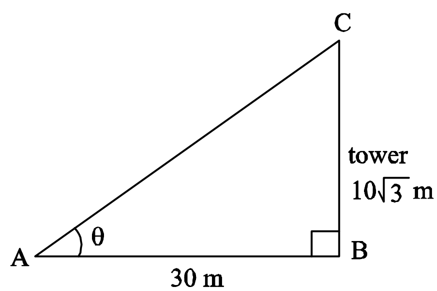
$$x = 12$$

$\therefore$ width of the road $= x + x$

$= 12 + 12$

$= 24$ m

1. Find the angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of a tower of height $10\sqrt{3}$ m.



$BC = 10\sqrt{3}$ m height of the tower.

$AB = 30$ m distance from foot of the tower.

To Find $\angle BAC = \theta$

In ΔABC

$$\tan\theta = \frac{BC}{AB}$$

$$= \frac{10\sqrt{3}}{30}$$

$$= \frac{\sqrt{3}}{3}$$

$$= \frac{\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

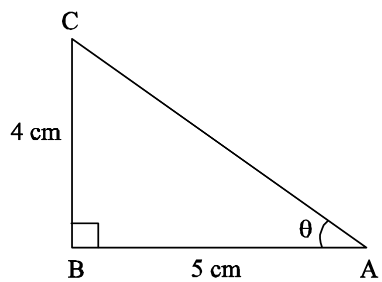
$$\theta = 30°$$

$\therefore$ Angle of elevation is 30°

Example 6.18

Calculate the size of $\angle BAC$ in the given triangles.

(i)



In ΔABC

$$\tan\theta = \frac{BC}{AB}$$

$$= \frac{4}{5}$$

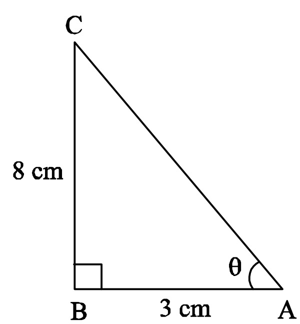
$$\tan\theta = 0.8$$

$$\theta = \tan^{-1}(0.8)$$

$$\theta = 38.7° \qquad \text{(from table)}$$

$$\therefore \angle BAC = 38.7°$$

(ii)



In ΔABC

$$\tan\theta = \frac{BC}{AB}$$

$$= \frac{8}{3}$$

$$\tan\theta = 2.66$$

$$\theta = \tan^{-1}(2.66)$$

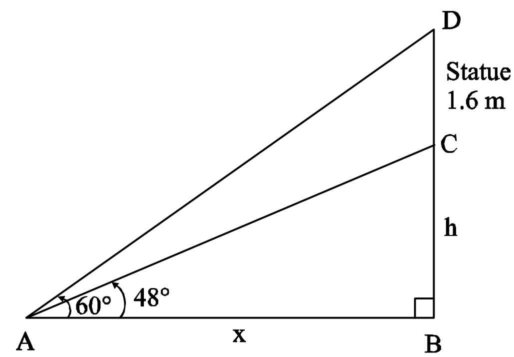
$$\theta = 69.4° \qquad \text{(from the table)}$$

$$\therefore \angle BAC = 69.4°$$

Type II: Angle of elevation (Two triangle combined sums)

Q.No: 4, Example 6.22, 3, 6, Example 6.23, 5, Example 6.21, 6.25, 7, 8, Example 6.24

4. A statue 1.6 m tall stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 40°. Find the height of the pedestal. ($\tan 40° = 0.8391$, $\sqrt{3} = 1.732$).



✍ **Solution:**

Let

BC = 'h' m height of the pedestal

CD = 1.6 m height of the statue

AB = 'x'

$\angle BAD = 60°$ and $\angle BAC = h$

- In ΔABC

$$\tan\theta = \frac{BC}{AB}$$

$$\tan 40° = \frac{h}{x}$$

$$0.8391 = \frac{h}{x}$$

$$x = \frac{h}{0.8391} \qquad ...(1)$$

- In ΔABD

$$\tan\theta° = \frac{BD}{AB}$$

$$\tan 60° = \frac{h+1.6}{x}$$

$$\sqrt{3} = \frac{h+1.6}{x}$$

$$x = \frac{h+1.6}{\sqrt{3}}$$

$$x = \frac{h+1.6}{1.732} \quad ...(2)$$

From (1) and (2)

$$\frac{h}{0.8391} = \frac{h+1.6}{1.732}$$

$$\frac{h}{0.84} = \frac{h+1.6}{1.7}$$

$$1.7h = 0.8h\,(h+1.6)$$

$$1.7h = 0.84h + 1.344$$

$$1.7h - 0.84h = 1.344$$

$$0.86h = 1.346$$

$$h = \frac{1.344}{0.86}$$

$$= \frac{134.4}{86}$$

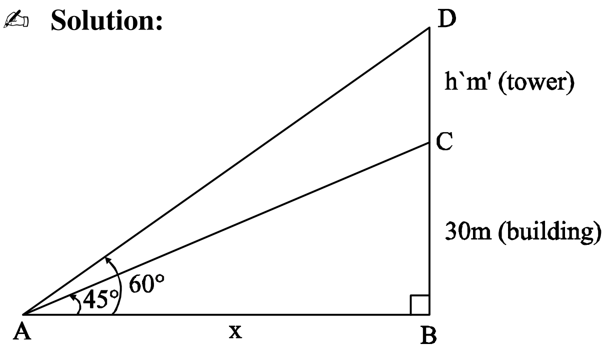
$h = 1.5$ m (app)

Height of the pedestal is 1.5 m

Example 6.22

From a point on the ground, the angles of elevation of the bottom and top of a tower fixed at the top of a 30 m high building are 45° and 60° respectively. Find the height of the tower, ($\sqrt{3} = 1.732$)

✍ **Solution:**



Let

$BC = 30$ m (height of the building)

$CD = 'h'$ m (height of the tower)

$AB = x$

$$\angle BAC = 45° \text{ and } \angle BAD = 60°$$

- In $\Delta\, BAC$

$$\tan\theta = \frac{BC}{AB}$$

$$\tan 45° = \frac{30}{x}$$

$$1 = \frac{30}{x}$$

$$\boxed{x = 30} \quad ...(1)$$

- In $\Delta\, BAD$

$$\tan\theta = \frac{BD}{AB}$$

$$\tan 60° = \frac{30+h}{x}$$

$$\sqrt{3} = \frac{30+h}{30} \quad \text{(From (1))}$$

$$30 + h = 30\sqrt{3}$$

$$h = 30\sqrt{3} - 30$$

$$h = 30(\sqrt{3} - 1)$$

$$= 30(1.732 - 1)$$

$$= 30 \times 0.732$$

$$h = 21.96 \text{ m}$$

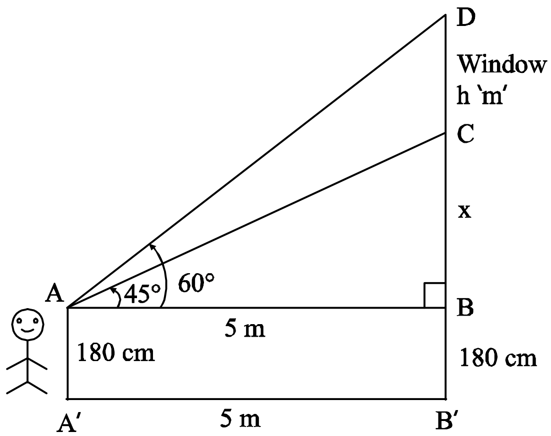
$\therefore$ Height of the tower is 21.96 m.

3. To a man standing outside his house, the angles of elevation of the top and bottom of a window are 60° and 45° respectively. If the height of the man is 180 cm and if he is 5 m away from the wall, what is the height of the window? ($\sqrt{3} = 1.732$).

✍ **Solution:**

Let

$CD = 'h'$ m height of the window



$AB = A'B' = 5$ m (distance from the wall)

BC = 'x' (height of the wall)

$AA' = BB' = 180$ cm (observer height)

$\angle BAC = 45°$ and $\angle BAD = 60°$

- In ΔABC,

$$\tan\theta = \frac{BC}{AB}$$

$$\tan 45° = \frac{x}{5}$$

$$1 = \frac{x}{5}$$

$$\boxed{x = 5} \quad \text{...(1)}$$

- In ΔABD

$$\tan\theta = \frac{BD}{AB}$$

$$\tan 60° = \frac{x+h}{5}$$

$$\sqrt{3} = \frac{x+h}{5}$$

$$\sqrt{3} = \frac{5+h}{5} \quad \text{(From (1))}$$

$$5 + h = \sqrt{3} \times 5$$

$$h = 5\sqrt{3} - 5$$

$$h = 5(\sqrt{3} - 1)$$

$$= 5(1.732 - 1)$$

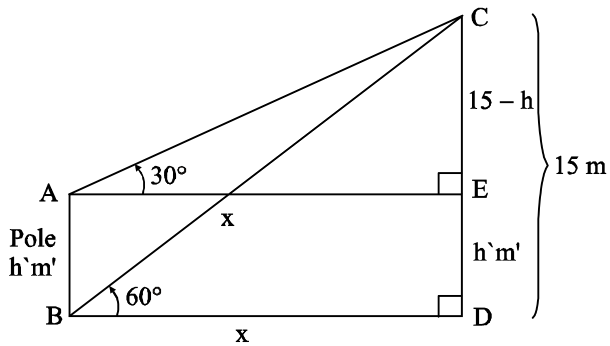
$$= 5 \times 0.732 = 3.660$$

$$\boxed{h = 3.66 \text{ m}}$$

$\therefore$ Height of the window is 3.66 m.

6. The top of a 15 m high tower makes an angle of elevation of 60° with the bottom of an electronic pole and angle of elevation of 30° with the top of the pole. What is the height of the electric pole?

✍ **Solution:**



Let

AB = 'h' m (height of the pole)

$CD = 15$ m (height of the tower)

Since DE = 'h' m then $CE = 15 - h$

Let $AE = BD = x$

$\angle DBC = 60°$ and $\angle EAC = 30°$

- In ΔBDC

$$\tan\theta = \frac{CD}{BD}$$

$$\tan 60° = \frac{15}{x}$$

$$\sqrt{3} = \frac{15}{x}$$

$$x = \frac{15}{\sqrt{3}} \quad \text{...(1)}$$

- In ΔAEC,

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 30° = \frac{15-h}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{15-h}{x}$$

$$x = \sqrt{3}(15 - h) \quad \text{...(2)}$$

From (1) and (2)

$$\frac{15}{\sqrt{3}} = \sqrt{3}\,(15 - h)$$

$$15 = 3\,(15 - h)$$

$$15 = 45 - 3h$$

$$3h = 45 - 15$$

$$3h = 30$$

$$h = \frac{30}{3}$$

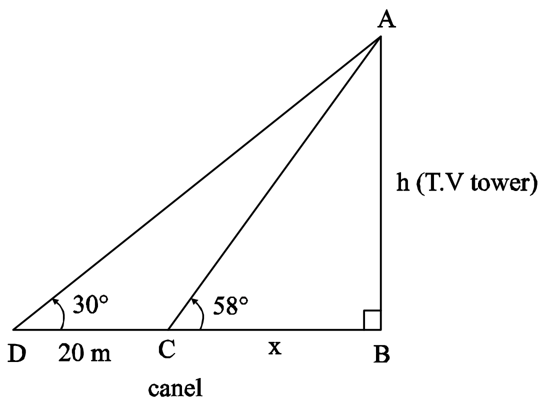
$$\boxed{h = 10 \text{ m}}$$

$\therefore$ Height of the electric pole is 10 m.

Example 6.23

A TV tower stands vertically on a bank of a canal. The tower is watched from a point on the other bank directly opposite to it. The angle of elevation of the top of the tower is 58°. From another point 20 m away from this point on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal. **($\tan 58° = 1.6003$)**

✍ **Solution:**



Let

AB = 'h' m (height of the T.V tower)

$CD = 20$ m

BC = 'x' m (width of the canal)

$\angle BCA = 58°$ and $\angle BDA = 30°$

- In ΔABC,

$$\tan\theta = \frac{AB}{BC}$$

$$\tan 58° = \frac{h}{x}$$

$$1.6003 = \frac{h}{x}$$

$$h = 1.6003x \qquad \text{...(1)}$$

- In ΔABD

$$\tan\theta = \frac{AB}{BD}$$

$$\tan 30° = \frac{h}{20 + x}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{20 + x}$$

$$\frac{20 + x}{\sqrt{3}} = h \qquad \text{...(2)}$$

From (1) and (2)

$$1.6003x = \frac{20 + x}{\sqrt{3}}$$

$$1.6003x \times \sqrt{3} = 20 + x$$

$$(1.6003x)\,(1.732) = 20 + x$$

$$2.77x - x = 20$$

$$1.77x = 20$$

$$x = \frac{20}{1.77}$$

$$x = \frac{2000}{177}$$

$$\boxed{x = 11.2 \text{ m}}$$

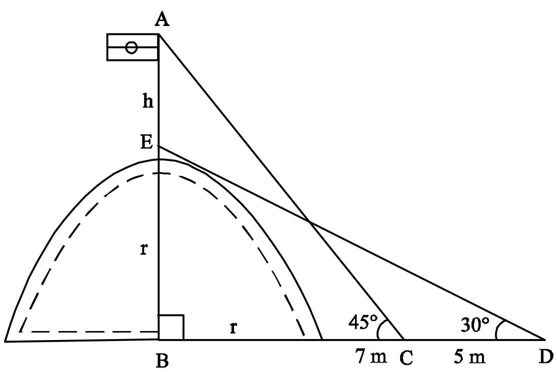
$$(1) \Rightarrow\ h = 1.6003 \times 11.2$$

$$\boxed{h = 17.9}$$

$\therefore$ Height of the tower is 17.9 m and width of the canal is 11.2 m.

5. A flag pole 'h' metres is on the top of the hemispherical dome of radius 'r' metres. A man is standing 7 m away from the dome. Seeing the top of the pole at an angle 45° and moving 5 m away from the dome and seeing the bottom of the pole at an angle 30°. Find (i) the height of the pole (ii) radius of the dome. ($\sqrt{3} = 1.732$)

✍ **Solution:**



Let

$AE =$ 'h' m (height of the pole)

$BE =$ 'r' (radius of the dome)

Given $BC = 7$ m, $CD = 5$ m

$\angle ACB = 45°$ and $\angle EDB = 30°$

- In ΔABC

$$\tan\theta = \frac{O.S}{A.S}$$

$$\tan 45° = \frac{h+r}{7}$$

$$1 = \frac{h+r}{r+7}$$

$$h + r = r + 7 \quad \text{...(1)}$$

$$\boxed{h = 7 \text{ m}}$$

- In ΔBDE

$$\tan\theta = \frac{BE}{BD}$$

$$\tan 30° = \frac{r}{7+5+r}$$

$$\frac{1}{\sqrt{3}} = \frac{r}{7+5+r}$$

$$\frac{1}{\sqrt{3}} = \frac{r}{12+r}$$

$$\sqrt{3}\, r = 12 + r$$

$$\sqrt{3}\, r - r = 12$$

$$r(\sqrt{3} - 1) = 12$$

$$r = \frac{12}{\sqrt{3}-1}$$

$$r = \frac{12}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{12(\sqrt{3}+1)}{(\sqrt{3})^2 - (1)^2}$$

$$= \frac{12(1.732+1)}{3-1}$$

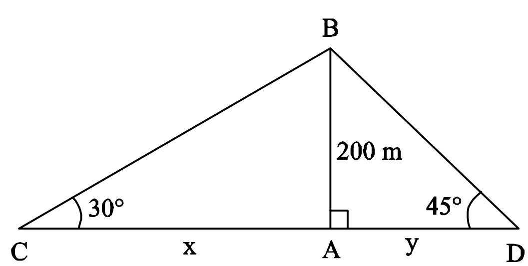
$$= \frac{12 \times 2.732}{2}$$

$$= 6 \times 2.732$$

$$\boxed{r = 16.392 \text{ m}}$$

Example 6.21

Two ships are sailing in the sea on either sides of a lighthouse. The angle of elevation of the top of the lighthouse as observed from the ships are 30° and 45° respectively. If the lighthouse is 200 m high, find the distance between the two ships. ($\sqrt{3} = 1.732$).



✍ **Solution:**

Let $AB = 200$ m height of the light house.

$CD = (x + y)$ distance between two ships

$$\angle ACB = 30° \text{ and } \angle ADB = 45°$$

- In ΔBAD

$$\tan 45° = \frac{200}{y}$$

$$1 = \frac{200}{y}$$

$$\boxed{y = 200 \text{ m}}$$

- In ΔBAC

$$\tan 30° = \frac{200}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{200}{x}$$

$$x = 200\sqrt{3}$$

$$= 200 \times 1.732$$

$$\boxed{x = 346.4 \text{ m}}$$

$\therefore$ Distance between two ships

$$CD = 346.4 + 200$$

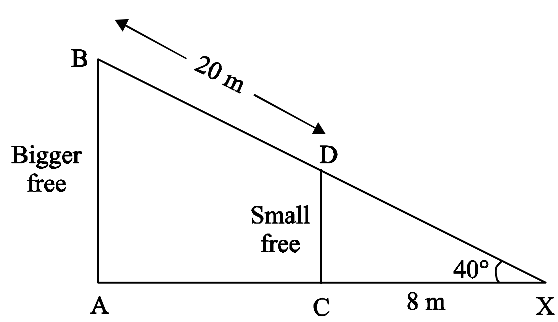
$$= 546.4 \text{ m}$$

Example 6.25

Two trees are standing on flat g round. The angle of elevation of the top of both the trees from a point X on the ground is 40°. If the horizontal distance between X and the smaller tree is 8 m and the distance of the top of the two trees is 20 m, calculate

(i) the distance between the point X and the top of the smaller tree.

(ii) the horizontal distance between the two trees. **(cos 40° = 0.7660)**



$$AB = \text{bigger tree}$$

$$CD = \text{smaller tree}$$

X is the point on the ground.

$$BD = 20 \text{ m}$$

$$\angle A \times B = \angle C \times D = 40°$$

To find: $XD = ?, AC = ?$

- In $\Delta \times CD$

$$\cos 40° = \frac{8}{XD}$$

$$0.7660 = \frac{8}{XD}$$

$$XD = \frac{8}{0.7660}$$

$$= \frac{8000}{766}$$

$$\boxed{XD = 10.44 \text{ m}}$$

- In ΔBAX

$$\cos 40° = \frac{AC + CX}{BD + DX}$$

$$0.7660 = \frac{AC + 8}{20 + 10.44}$$

$$0.7660 = \frac{AC + 8}{30.44}$$

$$AC + 8 = 0.7660 \times 30.44$$

$$AC = 23.32 - 8$$

$$\boxed{AC = 15.32 \text{ m}}$$

$\therefore$ (i) Distance between the point X and top of the small tree is 10.44 m.

(ii) The horizontal distance between two trees is 15.32 m.

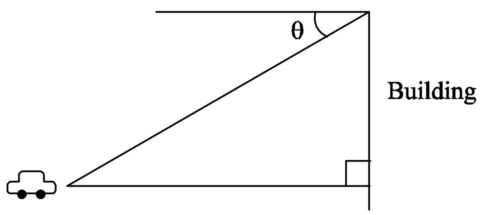
Exercise 6.3

KEY POINTS

1. Angle of depression

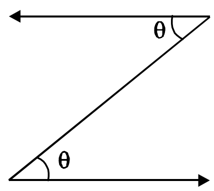
The angle of depression is an angle formed by the line of sight with the horizontal when the point is below the horizontal level.

In this case we move our head downwards to look at the object.



Note

Angle of depression and angle of elevation are equal become they are alternative angles.

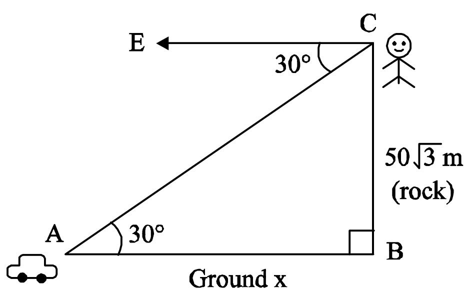


Type I: Angle of depression Given one angle.

Q.No.1, 2, Example 6.26, 6.27

1. From the top of a rock $50\sqrt{3}$ m high, the angle of depression of a car on the ground is observed to be 30°. Find the distance of the car from the rock.

✍ **Solution:**



$BC = 50\sqrt{3}$ m (Height of the rock)

$AB =$ 'x' m (distance of the car from the rock)

$\angle ACE = \angle BAC = 30°$

- In ΔABC,

$$\tan\theta = \frac{BC}{AB}$$

$$\tan 30° = \frac{50\sqrt{3}}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{50\sqrt{3}}{x}$$

$$x = 50 \times 3$$

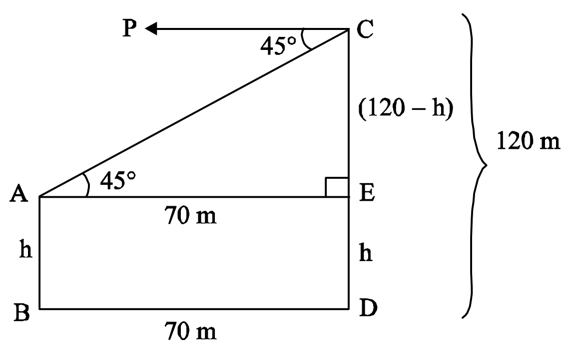
$$x = 150 \text{ m}$$

$\therefore$ The distance of the car from the rock is 150 m.

---

2. The horizontal distance between two buildings is 70 m. The angle of depression of the top of the first building when seen from the top of the second building is 45°. If the height of the second building is 120 m, find the height of the first building.

✍ **Solution:**



$AB =$ 'h' m (height of the 1st building)

$CD =$ 120 m (height of the 2nd building)

Since $DE = h$, we get $CE = (120 - h)$

$$\angle ACF = \angle EAC = 45°$$

- In ΔAEC

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 45° = \frac{120 - h}{70}$$

$$1 = \frac{120 - h}{70}$$

$$120 - h = 70$$

$$120 - 70 = h$$

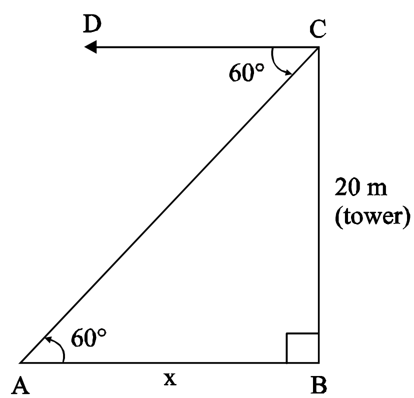
$$50 \text{ m} = h$$

$\therefore$ Height of the first building is 50 m.

Example 6.26

A player sitting on the top of a tower of height 20 m observes the angles of depression of a ball lying on the ground as 60°. Find the distance between the foot of the tower and the ball. ($\sqrt{3} = 1.732$)

✍ **Solution:**



$BC = 20$ m (tower)

$AB = $ 'x' m

$\angle ACD = 60°$ and $\angle BAC = 60°$

- In ΔABC

$$\tan\theta = \frac{BC}{AB}$$

$$\tan 60° = \frac{20}{x}$$

$$\sqrt{3} = \frac{20}{x}$$

$$x = \frac{20}{\sqrt{3}}$$

$$= \frac{20}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{20\sqrt{3}}{3}$$

$$= \frac{20 \times 1.732}{3}$$

$$= \frac{34.64}{3}$$

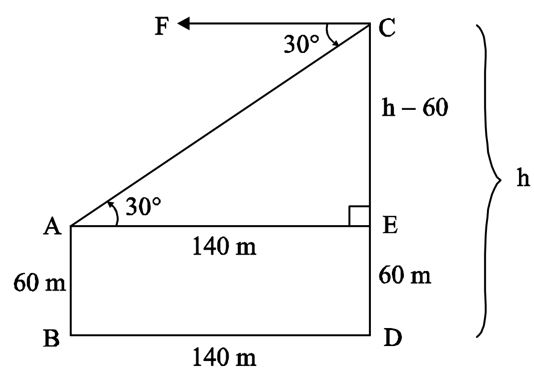
$$x = 11.54 \text{ m}$$

$\therefore$ distance between the foot of the tower and the ball is 11.54 m.

Example 6.27

The horizontal distance between two buildings is 140 m. The angle of depression of the top of the first building when seen from the top of the second building is 30°. If the height of the first building is 60 m, find the height of the second building. ($\sqrt{3} = 1.732$)

✍ **Solution:**



$AB = 60$ m (height of the 1st building)

$CD = h$ m (height of the 2nd building)

Since $DE = 60$ m then $CE = h - 60$

$\angle ACF = \angle EAC = 30°$

- In ΔAEC

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 30° = \frac{h - 60}{140}$$

$$\frac{1}{\sqrt{3}} = \frac{h-60}{140}$$

$$\sqrt{3}\,(h-60) = 140$$

$$h - 60 = \frac{140}{\sqrt{3}}$$

$$= \frac{140}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{140\sqrt{3}}{3}$$

$$= \frac{140 \times 1.732}{3}$$

$$= \frac{242.48}{3}$$

$$h - 60 = 80.82$$

$$h = 80.82 + 60$$

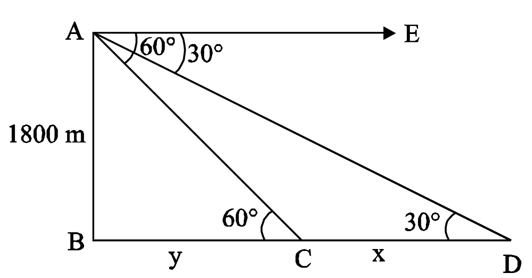
$$h = 140.82 \text{ m}$$

$\therefore$ Height of the 2nd building is 140.82 m

Type II: Angle of depression based sums (Given two angle)

Q.No: 4, Example 6.29, 6.30, 6.28, 3, 6 and 5.

4. An aeroplane at an altitude of 1800 m finds that two boats are sailing towards it in the same direction. The angles of depression of the boats as observed from the aeroplane are 60° and 30° respectively. Find the distance between the two boats. ($\sqrt{3} = 1.732$)



$AB = 1800$ m (Height of the aeroplane)

$CD = x$' m (distance between two boats)

$BC =$ 'y'

$\angle CAE = \angle ACB = 60°$ and

$\angle DAE = \angle ADB = 30°$

- In ΔABC

$$\tan\theta = \frac{AB}{BC}$$

$$\tan 60° = \frac{1800}{y}$$

$$\sqrt{3} = \frac{1800}{y}$$

$$y = \frac{1800}{\sqrt{3}} \quad ...(1)$$

- In ΔABD

$$\tan\theta = \frac{AB}{BD}$$

$$\tan 30° = \frac{1800}{x+y}$$

$$\frac{1}{\sqrt{3}} = \frac{1800}{x+y}$$

$$x + y = 1800\sqrt{3}$$

$$y = 1800\sqrt{3} - x \quad ...(2)$$

From (1) and (2)

$$\frac{1800}{\sqrt{3}} = 1800\sqrt{3} - x$$

$$1800 = 1800 \times 3 - \sqrt{3}x$$

$$1800 = 5400 - \sqrt{3}x$$

$$\sqrt{3}x = 5400 - 1800$$

$$x = \frac{3600}{\sqrt{3}}$$

$$= \frac{3600}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{3600\sqrt{3}}{3}$$

$$= 1200\sqrt{3}$$

$$= 1200 \times 1.732$$

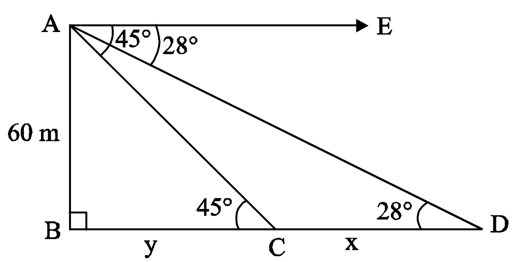
$$x = 2078.4$$

$\therefore$ Distance between two boats is 2078.4 m

Example 6.29

As observed from the top of a 60 m high light house from the sea level, the angles of depression of two ships are 28° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. **($\tan 28° = 0.5317$)**

✍ **Solution:**



$AB = 60$ m (Height of the light house)

$CD =$ 'x' m (distance between two ships)

$BC =$ 'y'

$\angle CAE = \angle ACB = 45°$

$\angle DAE = \angle ADB = 28°$

- In ΔABC

$$\tan\theta = \frac{AB}{BC}$$

$$\tan 45° = \frac{60}{y}$$

$$1 = \frac{60}{y}$$

$$\boxed{y = 60 \text{ m}} \quad \text{...(1)}$$

- In ΔABD

$$\tan\theta = \frac{AB}{BD}$$

$$\tan 28° = \frac{60}{x+y}$$

$$0.5317 = \frac{60}{x+y}$$

$$x + y = \frac{60}{0.5317}$$

$$x + 60 = \frac{600000}{5317} \quad \text{(From (1))}$$

$$x + 60 = 112.85$$

$$x = 112.85 - 60$$

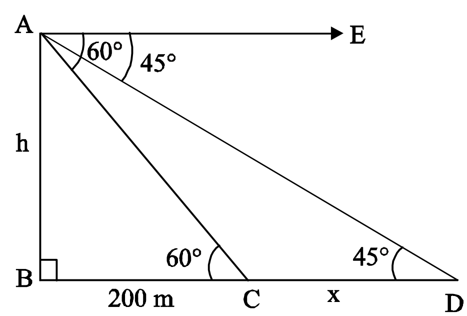
$$x = 52.85 \text{ m}$$

$\therefore$ Distance two ships is 52.85 m.

Example 6.30

A man is watching a boat speeding away from the top of a tower. The boat makes an angle of depression of 60° with the man's eye when at a distance of 200 m from the tower. After 10 seconds, the angle of depression become 45°. What is the approximate speed of the boat (in km/hr), assuming that it is sailing in still water? **($\sqrt{3} = 1.732$)**

✍ **Solution:**



$AB =$ 'h' m (height of the tower)

$CD =$ 'x' m (distance between two boats)

$BC = 200$ m

$\angle CAE = \angle ACB = 60°$ and

$\angle DAE = \angle ADB = 45°$

- In ΔABC

$$\tan\theta = \frac{AB}{AC}$$

$$\tan 60° = \frac{h}{200}$$

$$\sqrt{3} = \frac{h}{200}$$

$$h = 200\sqrt{3}$$

$$= 200 \times 1.732$$

$$\boxed{h = 346.4 \text{ m}} \quad ...(1)$$

- In $\Delta\, ABD$

$$\tan\theta = \frac{AB}{BD}$$

$$\tan 45° = \frac{h}{200+x}$$

$$1 = \frac{346.4}{200+x} \quad \text{(From (1))}$$

$$200 + x = 346.4$$

$$x = 346.4 - 200$$

$$\boxed{x = 146.4 \text{ m}}$$

Given

Distance covered in 10 seconds

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{146.4}{10}$$

$$= 14.64 \text{ m/s}$$

$$= 14.64 \times \frac{18}{5}$$

[m/s to km/m multiply by $\frac{3600}{1000} = \frac{18}{5}$

$$= \frac{263.52}{5}$$

$$= 52.704 \text{ km/hr}$$

$\therefore$ Speed of the boat is 52.704 km/hr

Example 6.28

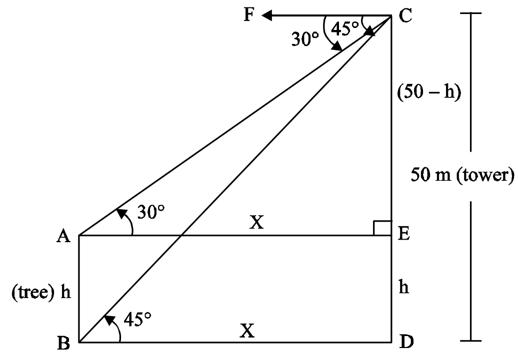
From the top of a tower 50 m high, the angles of depression of the top and bottom of a tree are observed to be 30° and 45° respectively. Find the height of the tree. **($\sqrt{3} = 1.732$)**

✍ **Solution:**

$CD = 50$ m (height of the tower)

$AB =$ 'h' m (height of the tree)

Since $DE =$ 'h' we get $CE = (50 - h)$



$$AE = BD = x$$

$$\therefore \angle ACF = \angle CAE = 30° \text{ and}$$

$$\angle BCF = \angle CBD = 45°$$

- In $\Delta\, BDC$

$$\tan\theta = \frac{CD}{BD}$$

$$\tan 45° = \frac{50}{x}$$

$$l = \frac{50}{x}$$

$$\boxed{x = 50} \quad ...(1)$$

- In $\Delta\, AEC$

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 30° = \frac{50-h}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{50-h}{50} \quad \text{(From (1))}$$

$$\sqrt{3}\,(50-h) = 50$$

$$50 - h = \frac{50}{\sqrt{3}}$$

$$50 - \frac{50}{\sqrt{3}} = h$$

$$\frac{50\sqrt{3} - 50}{\sqrt{3}} = h$$

$$\frac{50(\sqrt{3}-1)}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = h$$

$$\frac{50(3-\sqrt{3})}{3} = h$$

$$\frac{50\,(3-1.732)}{3} = h$$

$$\frac{50 \times 1.268}{3} = h$$

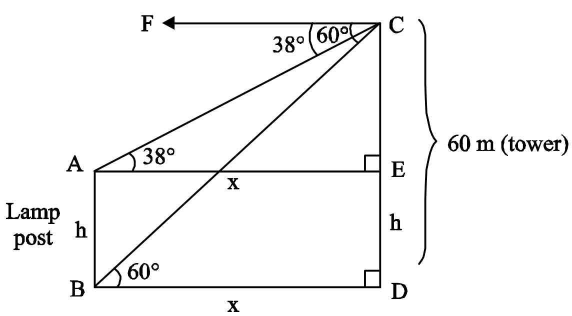
$$\frac{63.4}{3} = h$$

$$21.13 = h$$

$\therefore$ Height of the tree is 21.13 m.

---

3. From the top of the tower 60 m high the angle of depression of the top and bottom of a vertical lamp post are observed to be 38° and 60°. Find the height of the lamp post $\tan 38° = 0.7813$, $\sqrt{3} = 1.732$



✍ **Solution:**

$CD = 60$ m (Height of the tower)

$AB = $ 'h' m (Height of the lamp post)

$BD = AE = x$

$\angle ACF = \angle CAE = 38°$ and

$\angle BCF = \angle CBD = 60°$

- In $\Delta\, BDC$

$$\tan\theta = \frac{CD}{BD}$$

$$\tan 60° = \frac{60}{x}$$

$$\sqrt{3} = \frac{60}{x}$$

$$x = \frac{60}{\sqrt{3}}$$

$$= \frac{60}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{60\sqrt{3}}{3}$$

$$= 20\sqrt{3}$$

$$= 20 \times 1.732$$

$$\boxed{x = 34.64 \text{ m}}$$

- In $\Delta\, AEC$

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 38° = \frac{CE}{x}$$

$$0.7813 = \frac{CE}{34.64}$$

$$CE = 6.7813 \times 34.64$$

$$CE = 27.06$$

$\therefore$ Height of the lamp post $\Rightarrow CE + h = 60$

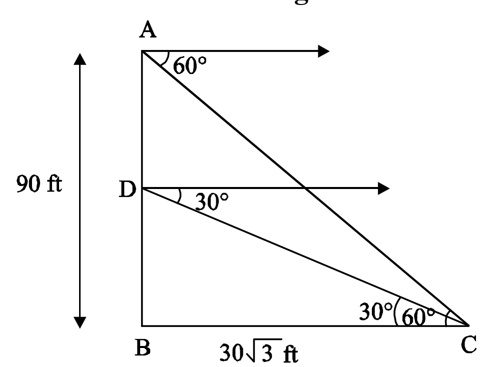
$$h = 60 - CE$$

$$h = 60 - 27.06$$

$$\boxed{h = 32.94 \text{ m}}$$

---

6. A lift in a building of height 90 feet with transparent glass walls is descending from the top of the building. At the top of the building, the angle of depression to a fountain in the garden is 60°. Two minutes later, the angle of depression reduces to 30°. If the fountain is $30\sqrt{3}$ feet from the entrance of the lift, find the speed of the lift which is descending.



$AB = 90$ ft (height of the lift)

'C' be the fountain

$BC = 30\sqrt{3}$ ft

- In $\Delta\, DBC$

$$\tan\theta = \frac{DB}{BC}$$

$$\tan 30° = \frac{DB}{30\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} = \frac{DB}{30\sqrt{3}}$$

$$\boxed{DB = 30 \text{ ft}}$$

$$\therefore AD = AB - DB$$

$$= 90 - 30$$

$$\boxed{AD = 60 \text{ ft}}$$

Distance of the lift moving descending from the top $= 60$ ft

$$\text{time} = 2 \text{ min}$$

$$\therefore \text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$= \frac{60}{2}$$

$$= 30 \text{ ft/min}$$

196.85 ft/min $= 1$ m/s

$$30 \text{ ft/min} = \frac{30}{196.85}$$

$$= \frac{3000}{19685}$$

$$= 0.1524 \text{ m/s}$$

Speed of the lift $= 0.1524$ m/s

---

5. From the top of a lighthouse, the angle of depression of two ships on the opposite sides of it are observed to be 30° and 60°. If the height of the lighthouse is h meters and the line joining the ships passes through the foot of the lighthouse, show that the distance between the ship is $\frac{4h}{\sqrt{3}}$ m.

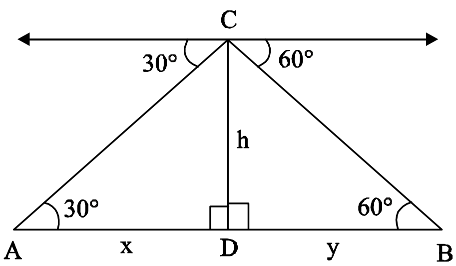
✍ **Solution:**

- $CD =$ 'h' m (height of the light house)

$AD = x$, $BD = y$

$AB = x + y$ (distance between two ships)

$\angle CAD = 30°$ and $\angle CBD = 60°$



- In $\Delta\, ADC$

$$\tan\theta = \frac{CD}{AD}$$

$$\tan 30° = \frac{h}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{x}$$

$$x = \sqrt{3}h \quad \text{...(1)}$$

- In $\Delta\, BDC$

$$\tan\theta = \frac{CD}{DB}$$

$$\tan 60° = \frac{h}{y}$$

$$\sqrt{3} = \frac{h}{y}$$

$$y = \frac{h}{\sqrt{3}} \quad \text{...(2)}$$

$$\therefore \text{Distance between two ships} = \sqrt{3}h + \frac{h}{\sqrt{3}}$$

$$= \frac{3h + h}{\sqrt{3}}$$

$$= \frac{4h}{\sqrt{3}} \text{ m}$$

Hence proved

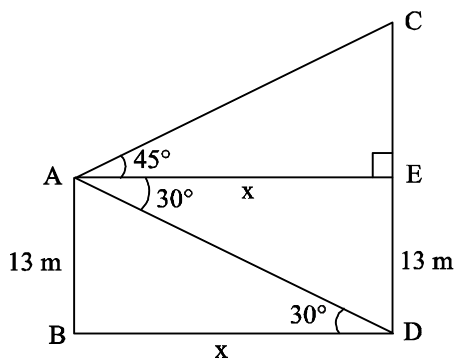
Exercise 6.4

Problems involving angle of elevation and depression:

Q.No: 1, 2, 4, 5 Example 6.31, 6.32, 6, 3, Example 6.33

1. From the top of a tree of height 13 m the angle of elevation and depression of the top and bottom of another tree are 45° and 30° respectively. Find the height of the top second tree. ($\sqrt{3} = 1.732$)

✍ **Solution:**



Let

$AB = 13$ m (height of 1^{st} tree)

$CD = CE + 13$ (height of 2^{nd} tree)

$AE = BD =$ 'x' m

$\angle CAE = 45°$ and $\angle EAD = 30°$

- In ΔAED

$$\tan\theta = \frac{DE}{AE}$$

$$\tan 30° = \frac{13}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{13}{x}$$

$$\boxed{x = 13\sqrt{3}\text{ m}}$$

- In ΔAEC,

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 45° = \frac{CE}{x}$$

$$1 = \frac{CE}{x}$$

$$x = CE$$

$$CE = 13\sqrt{3}\text{ m}$$

$\therefore$ Height of the 2^{nd} tree $= 13 + 13\sqrt{3}$

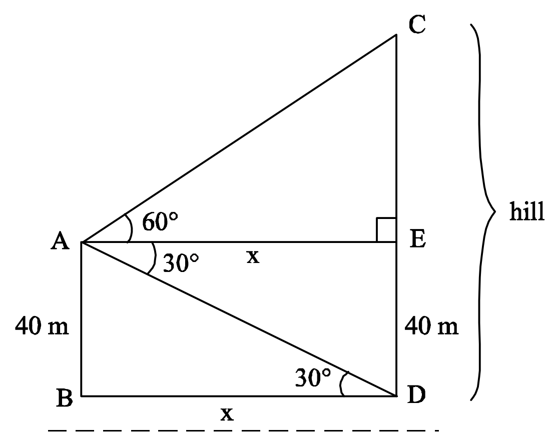
$= 13(1+\sqrt{3})$

$= 13(1+1.732)$

$= 35.52$ m

2. A man is standing on the dec of a ship, which is 40 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill. ($\sqrt{3} = 1.732$)

✍ **Solution:**



Let

$AB = 40$ m (deck of a ship)

$CD = CE + 40$ (Height of the hill)

$BD =$ 'x' distance of the hill from the ship

$AE =$ 'x'

$\angle CAE = 60°$ and $\angle DAE = 30°$

- In ΔAED

$$\tan\theta = \frac{DE}{AE}$$

$$\tan 30° = \frac{40}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{40}{x}$$

$$x = 40\sqrt{3} \quad ...(1)$$

$$= 40 \times 1.732$$

$$\boxed{x = 69.28 \text{ m}}$$

- In ΔAEC

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 60° = \frac{CE}{x}$$

$$\sqrt{3} = \frac{CE}{x}$$

$$CE = \sqrt{3}x$$

$$= \sqrt{3}\,(40\sqrt{3}) \quad \text{From (1)}$$

$$CE = 120 \text{ m}$$

$\therefore$ Height of the hill $= 40 + 120$

$= 160$ m

Distance of the hill from the ship $= 69.28$ m

---

4. The angle of elevation of the top of a cell phone tower from the foot of a high apartment is 60° and the angle of depression of the foot of the tower from the top of the apartment is 30°. If the height of the apartment is 50 m, find the height of the cell phone tower. According to radiations control norms, the minimum height of a cell phone tower should be 120 m. State if the height of the above mentioned cell phone tower meets the radiation norms.

✍ **Solution:**

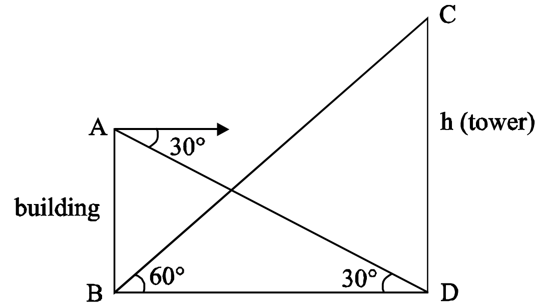
Let

$AB = 50$ m (height of the apartment)

$CD =$ 'h' m (height of the cell phone tower)

$\angle CBD = 60°$ and

$\angle ADB = 30°$



- In ΔABD

$$\tan\theta = \frac{AB}{BD}$$

$$\tan 30° = \frac{50}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{50}{x}$$

$$\boxed{x = 50\sqrt{3} \text{ m}} \quad ...(1)$$

- In ΔBDC

$$\tan\theta = \frac{CD}{BD}$$

$$\tan 60° = \frac{h}{x}$$

$$\sqrt{3} = \frac{h}{x}$$

$$\sqrt{3} = \frac{h}{50\sqrt{3}} \quad \text{(From (1))}$$

$$h = 50 \times 3$$

$$\boxed{h = 150 \text{ m}}$$

$\therefore$ Height of the cell phone tower is 150 m $>$ 120 m (Required height)

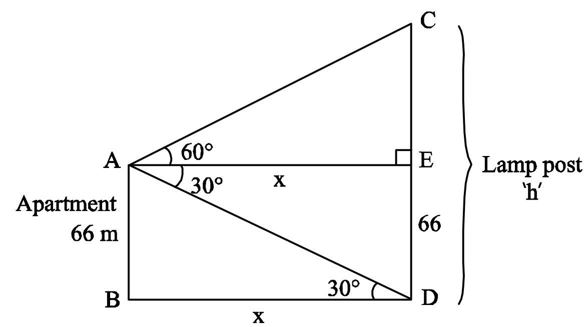
$\therefore$ The cell phone tower meets the radiation norms.

---

5. The angles of elevation and depression of the top and bottom of a lamp post from the top of a 66 m high apartment are 60° and 30° respectively. Find (i) The height of the lamp post.
(ii) The difference between height of the lamp post and the apartment.

(iii) The distance between the lamp post and the apartment. ($\sqrt{3} = 1.732$)

Solution:



Let

$AB = 66$ m (height of the apartment)

$CD =$ 'h' m $(CE + 66)$ (Height of the lamp post)

$AE = BD =$ 'x'

$\angle CAE = 60°$ and $\angle EAD = 30°$

- In ΔAED

$$\tan\theta = \frac{DE}{AE}$$

$$\tan 30° = \frac{66}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{66}{x}$$

$$x = 66\sqrt{3} \qquad ...(1)$$

$$x = 66 \times 1.732$$

$$x = 114.31 \text{ m}$$

- In ΔAEC,

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 60° = \frac{CE}{x}$$

$$\sqrt{3} = \frac{CE}{66\sqrt{3}} \qquad \text{From (1)}$$

$$CE = 66 \times 3$$

$$CE = 198$$

$\therefore$ height of the lamp post

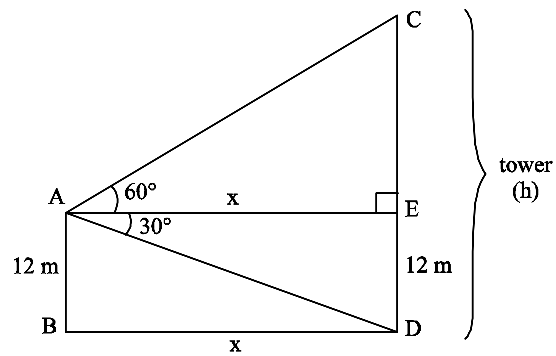
$$= 66 + 198$$

$$\boxed{h = 264 \text{ m}}$$

Example 6.31

From the top of a 12 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 30°. Determine the height of the tower.

Solution:



Let

$AB = 12$ m (Height of the tower)

$CD = CE + 12$ (Height of the cable tower)

$BD = AE = x$

$\angle CAE = 60°$ and $\angle DAE = 30°$

- In ΔAED

$$\tan\theta = \frac{DE}{AE}$$

$$\tan 30° = \frac{12}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{12}{x}$$

$$x = 12\sqrt{3} \qquad ...(1)$$

- In ΔAEC

$$\tan\theta = \frac{CE}{AE}$$

$$\tan 60° = \frac{CE}{x}$$

$$\sqrt{3} = \frac{CE}{x}$$

$$\sqrt{3} = \frac{CE}{12\sqrt{3}} \qquad \text{From (1)}$$

$$CE = 12 \times 3$$

$$= 36$$

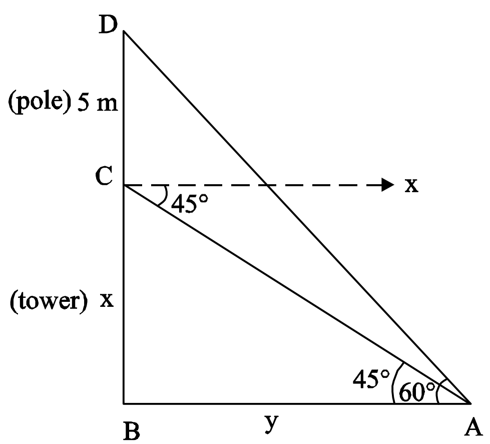
$\therefore$ Height of the cable tower

$$= 36 + 12 = 48 \text{ m}$$

Example 6.32

A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point 'A' on the ground is 60° and the angle of depression to the point 'A' from the top of the tower is 45°. Find the height of tower. **($\sqrt{3} = 1.732$)**

Solution:



Let

$CD = 5$ m (height of the pole)

$BC =$ 'x' (height of the tower)

$\angle BAD = 60°$ and $\angle XCA = \angle BAC = 45°$

- In ΔABC

$$\tan\theta = \frac{BC}{AB}$$

$$\tan 45° = \frac{x}{y}$$

$$1 = \frac{x}{y}$$

$$\boxed{x = y} \quad \text{...(1)}$$

- In ΔABD

$$\tan\theta = \frac{BD}{AB}$$

$$\tan 60° = \frac{x+5}{y}$$

$$\sqrt{3} = \frac{x+5}{x} \quad \text{From (1)}$$

$$\sqrt{3}x = x + 5$$

$$\sqrt{3}x - x = 5$$

$$x(\sqrt{3} - 1) = 5$$

$$x(\sqrt{3} - 1) = 5$$

$$x = \frac{5}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= \frac{5(\sqrt{3} + 1)}{((\sqrt{3})^2 - (1)^2}$$

$$= \frac{5(1.732 + 1)}{3 - 1}$$

$$= \frac{5 \times 2.732}{2}$$

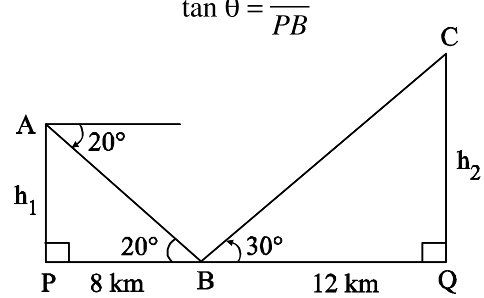
$$x = 6.83$$

$\therefore$ Height of the tower is 6.83 m.

6. Three villagers A, B and C can see each other across a valley. The horizontal distance between A and B is 8 km and the horizontal distance between B and C is 12 km. The angle of depression of B from A is 20° and the angle of elevation of C from B is 30°. Calculate: (i) the vertical height between A and B. (ii) the vertical height between B and C. ($\tan 20° = 0.3640$, $\sqrt{3} = 1.732$)

- In ΔAPB

$$\tan\theta = \frac{AP}{PB}$$



$$\tan 20° = \frac{h_1}{8}$$

$$0.3640 = \frac{h_1}{8}$$

$$h_1 = 0.3640 \times 8$$

$$= 29.12 \text{ km}$$

- In ΔBQC

$$\tan\theta = \frac{CQ}{BQ}$$

$$\tan 30° = \frac{h_2}{12}$$

$$\frac{1}{\sqrt{3}} = \frac{h_2}{12}$$

$$\sqrt{3}h_2 = 12$$

$$h_2 = \frac{12}{\sqrt{3}}$$

$$= \frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{12\sqrt{3}}{3}$$

$$= 4\sqrt{3}$$

$$= 4 \times 1.732$$

$$= 6.928$$

$$= 6.93$$

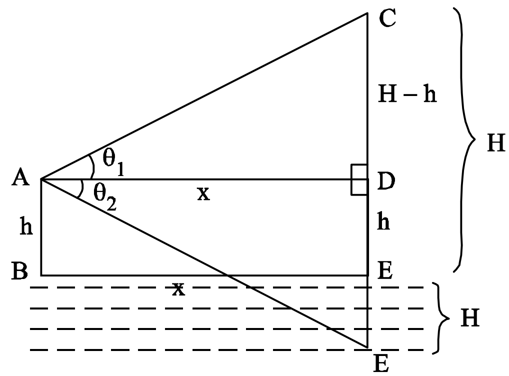
(i) The vertical height between A and B is 29.12 km

(ii) The vertical height between B and C is 6.93.

---

3. If the angle of elevation of a cloud from a point '*h*' meters above a lake is θ_1, and the angle of depression of its reflection in the lake is θ_2. Prove that the height that the cloud is located from the ground is $\dfrac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$

$AB = h$ height of the lake



$CE = EF =$ 'H' (Height of the cloud above the ground and reflection in water)

- as $DE = h$, we get

$CD = H - h$ and $DF = H + h$

we have to prove

$$H = \frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$$

- In ΔADC

$$\tan\theta_1 = \frac{H-h}{x}$$

$$x = \frac{H-h}{\tan\theta_1} \quad ...(1)$$

- In ΔDAF

$$\tan\theta_2 = \frac{H+h}{x}$$

$$x = \frac{H+h}{\tan\theta_2} \quad ...(12)$$

From (1) & (2)

$$\frac{H-h}{\tan\theta_1} = \frac{H+h}{\tan\theta_2}$$

$$(H-h)(\tan\theta_2) = (H+h)\tan\theta_1$$

$$H\tan\theta_2 - h\tan\theta_2 = H\tan\theta_1 + h\tan\theta_1$$

$$H\tan\theta_2 - H\tan\theta_1 = h\tan\theta_1 + h\tan\theta_2$$

$$H(\tan\theta_2 - \tan\theta_1) = h(\tan\theta_1 + \tan\theta_2)$$

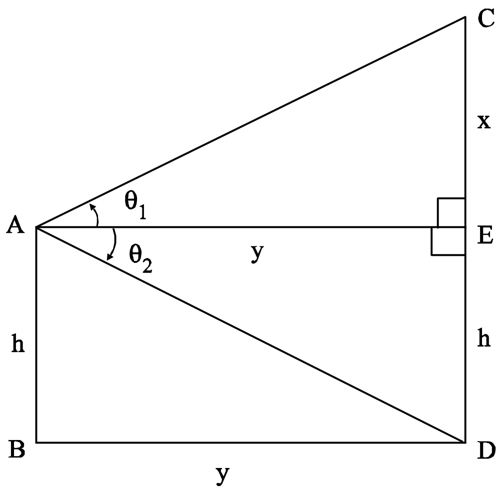
$$H = \frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$$

Proved.

Example 6.23

From a window (h meters high above the ground) of a house in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the street are θ_1 and θ_2 respectively. Show that the height of the opposite house is $h\left(1+\frac{\cot\theta_2}{\cot\theta_1}\right)$

Solution:



$AB =$ 'h'm (Height of the window)

$CD = (x+h)$ (Height of the house

$AE = BD = y$

- In ΔAEC

$$\tan\theta_1 = \frac{CE}{AE}$$

$$\tan\theta_1 = \frac{x}{y}$$

$$y = \frac{x}{\tan\theta_1}$$

$$y = x\cot\theta_1 \quad ...(1)$$

- In ΔAED

$$\tan\theta = \frac{DE}{AE}$$

$$\tan\theta_2 = \frac{h}{y}$$

$$y = \frac{h}{\tan\theta_2}$$

$$y = h\cot\theta_2 \quad ...(2)$$

From (1) & (2)

$$x\cot\theta_1 = h\cot\theta_2$$

$$x = \frac{h\cot\theta_2}{\cot\theta_1}$$

$\therefore$ Height of the house $= x+h$

$$= \frac{h\cot\theta_2}{\cot\theta_1} + h$$

$$= h\left(1+\frac{\cot\theta_2}{\cot\theta_1}\right) \text{proved}$$

Exercise 6.5

Multiple Choice Questions

1. The value of $\sin^2\theta + \frac{1}{1+\tan^2\theta}$ is equal to

1, $\tan^2\theta$ 2. 1 3. $\cot^2\theta$ 4. 0

Solution:

$$\sin^2\theta + \frac{1}{1+\tan^2\theta}$$

$$= \sin^2\theta + \frac{1}{\sec^2\theta}$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

Ans. (2) 1

---

2. $\tan\theta\,\mathrm{cosec}^2\theta - \tan\theta$ is equal to

1. $\sec\theta$ 2. $\cot^2\theta$
3. $\sin\theta$ 4. $\cot\theta$

Solution:

$$\tan\theta\,\mathrm{cosec}^2\theta - \tan\theta$$

$$= \tan\theta\,(\mathrm{cosec}^2\theta - 1)$$

$$= \tan\theta\cot^2\theta$$

CHAPTER 7

MENSURATION

Exercise 7.1

KEY POINTS

Surface Area

Surface Area is the measurement of all exposed area of a solid object.

I. Right circular cylinder

A right circular cylinder is a solid generated by the revolution of a Rectangle about of its sides as axis.

1. Curved surface Area

C.S.A = base perimeter × height

$$\boxed{\textbf{C.S.A} = 2\pi r h \text{ sq.u}}$$

2. Total surface Area

$$\text{T.S.A} = \text{C.S.A} + 2 \text{ base area}$$

$$= 2\pi r h + 2\pi r^2$$

$$\boxed{\textbf{T.S.A} = 2\pi r (h + r) \text{ sq.u}}$$

Note:

base of the cylinder – circle shape

- base perimeter $= 2\pi r$ units
- base Area $= \pi r^2$ sq.units

II. Hollow cylinder

An object bounded by two co-axial cylinders of the same height and different radii is called a "hollow cylinder".

1. $\text{C.S.A} = \text{outer C.S.A} + \text{inner C.S.A}$

$$= 2\pi R h + 2\pi r h$$

$$\boxed{\textbf{C.S.A} = 2\pi h (R + r) \text{ sq. units}}$$

2. $\text{T.S.A} = \text{C.S.A} + 2 \text{ base area}$

$$= 2\pi h (R + r) + 2\pi (R^2 - r^2)$$

$$= 2\pi h (R + r) + 2\pi (R + r)(R - r)$$

$$\text{T.S.A} = 2\pi (R + r)(R - r + h) \text{ sq. units}$$

Note:

base of the hollow cylinder is circular ring or path

- Area of circular path $= \pi (R^2 - r^2)$ sq. u

III. Right circular cone

A right circular cone is a solid generated by the revolution of a right angled triangle about one of the sides containing the right angle as axis.

1. $\text{C.S.A} = \frac{1}{2} \times \text{base perimeter} \times \text{slant height}$

$$= \frac{1}{2} \times 2\pi r \times l$$

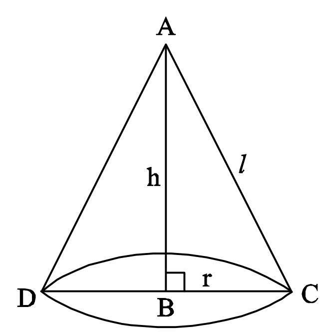
$$\boxed{\textbf{C.S.A} = \pi r l \text{ sq. u}}$$

2. $\text{T.S.A} = \text{C.S.A} + \text{base Area}$

$$= \pi r l + \pi r^2$$

$$\boxed{\textbf{T.S.A} = \pi r (l + r) \text{ sq. units}}$$

Note:



$$l = \sqrt{h^2 + r^2}$$

$$r = \sqrt{l^2 - h^2}$$

$$h = \sqrt{l^2 - r^2}$$

IV. The sphere

A sphere is a solid generated by the revolution of a semi circle about its diameter as axis.

Surface area of a sphere $= \mathbf{4\pi r^2}$ **sq u.**

V. Hemisphere

A section of the sphere cut by a plane through any of its great circle is a hemisphere.

1. C.S.A $= 2\pi r^2$ sq. u
2. T.S.A $=$ C.S.A + base area

$$= 2\pi r^2 + \pi r^2$$

$$= 3\pi r^2 \text{ sq. u.}$$

VI. Hollow Hemisphere

1. C.S.A $=$ outer C.S.A + inner C.S.A

$$= 2\pi R^2 + 2\pi r^2$$

$$\boxed{\textbf{C.S.A} = 2\pi (R^2 + r^2) \textbf{ sq. u}}$$

2. T.S.A $=$ C.S.A + base area

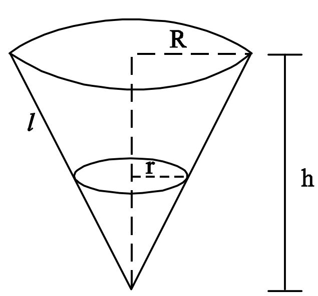
$$= 2\pi (R^2 + r^2) + \pi (R^2 - r^2)$$

$$= \pi [2R^2 + 2r^2 + R^2 - r^2]$$

$$\boxed{\textbf{T.S.A} = \pi [3R^2 + r^2] \textbf{ sq. u}}$$

VII. Frustrum of a right circular cone

When a cone cut through by a plane parallel to its base, the portion of the cone between the cutting plane and the base is called a frustum of the cone.



Here

$$l = \sqrt{h^2 + (R - r)^2}$$

1. $\text{C.S.A} = \frac{1}{2} \times (\text{sum of top and bottom perimeter}) \times l$

$$= \frac{1}{2} \times [2\pi R + 2\pi r]\, l$$

$$\boxed{\textbf{C.S.A} = \pi (R + r)\, l \textbf{ sq. u}}$$

2. T.S.A $=$ C.S.A + top area + bottom area

$$= \pi (R + r)\, l + \pi R^2 + \pi r^2$$

$$\boxed{\textbf{T.S.A} = \pi [(R + r)\, l + R^2 + r^2] \textbf{ sq. u}}$$

Note:

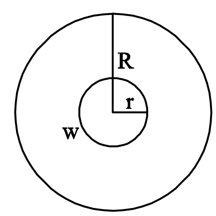
1. Cylinder, cone, Hemisphere and Frustum of cone having the circular base

- base area $= \pi r^2$ sq. u
- base perimeter $= 2\pi r$ units

2. Hollow cylinder, Hollow hemisphere having the circular path (or) ring base

- base area $= \pi (R^2 - r^2)$ sq. u
- base perimeter $= 2\pi (R + r)$ u

3. External radius (R), internal radius (r), thickness (w), then



- $W = R - r$
- $R = r + W$
- $r = R - W$

Type I: Problems based on cylinder Hollow cylinder

Example 7.1, 7.2, 1, 2, 7.3, 3, 7.4

Example 7.1

A cylindrical drum has a height of 20 cm and base radius of 14 cm. Find its curved surface area and the total surface area.

Cylinder: $r = 14$ cm

$h = 20$ cm

- C.S.A $= 2\pi r h$

$$= 2 \times \frac{22}{7} \times 14 \times 20$$

$= 2 \times 22 \times 2 \times 20$

$= 1760 \text{ cm}^2$

- T.S.A $= 2\pi r (h + r)$

$= 2 \times \frac{22}{7} \times 14 (20 + 14)$

$= 2 \times 22 \times 2 \times 34$

$= 2992 \text{ cm}^2$

Example 7.2

The curved surface area of a right circular cylinder of height 14 cm is 88 cm^2. Find the diameter of the cylinder.

Cylinder: $h = 14$ cm

C.S.A $= 88 \text{ cm}^2$

- C.S.A $= 88 \text{ cm}^2$

$$2\pi r h = 88$$

$$2 \times \frac{22}{7} \times r \times 14 = 88$$

$$r = \frac{88 \times 7}{2 \times 22 \times 14}$$

$$\boxed{r = 1 \text{ cm}}$$

$\therefore$ diameter $= 2r$

$= 2 (1)$

$= 2$ cm

---

1. The radius and height of a cylinder are in the ratio 5:7 and its curved surface area is 5500 sq.cm. Find its radius and height.

Cylinder

$r{:}h = 5{:}7$

C.S.A $= 5500 \text{ cm}^2$

Let $r = 5k$

$h = 7k$

- $2\pi r h = 5500 \text{ cm}^2$

$$2 \times \frac{22}{7} \times 5k \times 7k = 5500$$

$$220k^2 = 5500$$

$$k^2 = \frac{5500}{220}$$

$$k^2 = 25$$

$$k = 5$$

$\therefore r = 5 (5) = 25$ cm

$h = 7 (5) = 35$ cm

---

2. A solid iron cylinder has total surface area of 1848 sq.m. Its curved surface area is five - sixth of its total surface area. Find the radius and height of the iron cylinder.

Cylinder

T.S.A $= 1848$ sq.m

C.S.A $= \frac{5}{6}$ T.S.A

$= \frac{5}{6} \times 1848$

$= 5 \times 308$

C.S.A $= 1540 \text{ m}^2$

$$\boxed{\textbf{T.S.A = C.S.A + 2 base area}}$$

$$1848 = 1540 + 2\pi r^2$$

$$1848 - 1540 = 2\pi r^2$$

$$308 = 2 \times \frac{22}{7} \times r^2$$

$$\frac{308 \times 7}{2 \times 22} = r^2$$

$$7 \times 7 = r^2$$

$$\boxed{7m = r}$$

- C.S.A $= 1540$

$$2\pi r h = 1540$$

$$2 \times \frac{22}{7} \times 7 \times h = 1540$$

$$h = \frac{1540}{2 \times 22}$$

$$\boxed{h = 35 \text{ m}}$$

Example 7.3

A garden roller whose length is 3 m long and whose diameter is 2.8 m is rolled to level a garden. How much area will it cover in 8 revolutions?

Garden Roller (cylinder)

length $(h) = 3$ m

diameter $= 2.8$ m

$$r = \frac{2.8}{2}$$

$$r = 1.4 \text{ m}$$

1 Revolution $= \text{C.S.A}$

$$= 2\pi r h$$

$$= 2 \times \frac{22}{7} \times 1.4 \times 3$$

$$= 26.4 \text{ m}^2$$

8 Revolution $= 26.4 \times 8$

$$= 211.2 \text{ m}^2$$

---

3. The external radius and the length of a hollow wooden log are 16 cm and 13 cm respectively. If its thickness is 4 cm then find its T.S.A.

Hollow wooden log (hollow cylinder)

Given

$$\mathbf{R = 16 \text{ cm}}$$

$$\mathbf{h = 13 \text{ cm}}$$

$$\mathbf{w = 4 \text{ cm}}$$

$$r = R - W$$

$$= 16 - 4$$

$$r = 12 \text{ cm}$$

$$\text{T.S.A} = 2\pi (R + r)(R - r + h)$$

$$= 2 \times \frac{22}{7} (16 + 12)(16 - 12 + 13)$$

$$= 2 \times \frac{22}{7} \times 28 \times 17$$

$$= 2 \times 22 \times 4 \times 17$$

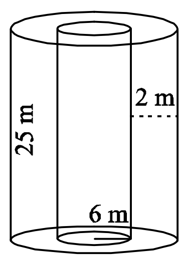
$$\text{T.S.A} = 2992 \text{ cm}^2$$

Example 7.4

If one litre of paint covers* 10 m^2, *how many litres of paint is required to paint the internal and external surface areas of a cylindrical tunnel whose thickness is 2 m, internal radius is 6 m and height is 25 m.

Cylindrical tunnel (Hollow cylinder)

Given



$r = 6$ m

$h = 25$ m

$W = 2$ m

$$\therefore R = r + W$$

$$= 6 + 2$$

$$R = 8 \text{ m}$$

$$\therefore \text{C.S.A} = 2\pi (R + r) h$$

$$= 2 \times \frac{22}{7} (8 + 6) \times 25$$

$$= 2 \times \frac{22}{7} \times 14 \times 25$$

$$= 2 \times 22 \times 2 \times 25$$

$$= 2200 \text{ m}^2$$

Area covered by one litre of paint $= 10 \text{ m}^2$

Number of litres required to paint the tunnel

$$= \frac{2200}{10}$$

$$= 220 \text{ litre}$$

Type II: Problems based on cone

Example 7.5, 7.6, 5, 6, 4, 7, Example 7.7

Example 7.5

The radius of a conical tent is 7 m and the height is 24 m. Calculate the length of the canvas used to make the tent if the width of the rectangular canvas is 4 m?

Conical tent

$$r = 7 \text{ m}$$

$$h = 24 \text{ m}$$

$$l = \sqrt{r^2 + h^2}$$

$$= \sqrt{(7)^2 + (24)^2}$$

$$= \sqrt{49 + 576}$$

$$= \sqrt{625}$$

$$\boxed{l = 25 \text{ m}}$$

C.S.A of the conical tent $= \pi r l$ sq. u

$$= \frac{22}{7} \times 7 \times 25$$

$$= 550 \text{ m}^2$$

Given

Width of the rectangular

Canvas $= 4$ m

$$l = ?$$

$$l \times b = \text{C.S.A}$$

$$l \times 4 = 550$$

$$l = \frac{550}{4}$$

$$l = 137.5 \text{ m}$$

$\therefore$ The length of the canvas 137.5 m

Example 7.6

If the total surface area of a cone of radius 7 cm is 704 cm^2, then find its slant height.

Cone

$$r = 7 \text{ cm}$$

$$\text{T.S.A} = 704 \text{ cm}^2$$

$$l = ?$$

- T.S.A $= 704 \text{ cm}^2$

$$\pi r (l + r) = 704$$

$$\frac{22}{7} \times 7 (l + 7) = 704$$

$$l + 7 = \frac{704}{22}$$

$$l + 7 = 32$$

$$l = 32 - 7$$

$$\boxed{l = 25 \text{ cm}}$$

5. **4 persons live in a conical tent whose slant height is 19 cm. If each person require 22 cm^2 of the floor area, then find the height of the tent.**

Conical tent

$l = 19$ cm

base Area of 1 person $= 22 \text{ cm}^2$

base Area of 4 persons $= 22 \times 4$

$$= 88 \text{ cm}^2$$

$$\pi r^2 = 88 \text{ cm}^2$$

$$\frac{22}{7} \times r^2 = 88$$

$$r^2 = 88 \times \frac{7}{22}$$

$$\boxed{r^2 = 28 \text{ cm}}$$

$$h = \sqrt{l^2 - r^2}$$

$$= \sqrt{19^2 - 28}$$

$$= \sqrt{361 - 28}$$

$$= \sqrt{333}$$

$$\boxed{h = 18.2 \text{ cm (app)}}$$

	18.2
1	3,33
	1
28	233
	224
362	900
	724
	176

6. A girl wishes to prepare birthday caps in the form of right circular cones for her birthday party, using a sheet of paper whose area is 5720 cm², how many caps can be made with radius 5 cm and height 12 cm.

Cap (cone)

$$r = 5 \text{ cm}$$
$$h = 12 \text{ cm}$$
$$\text{Paper Area} = 5720 \text{ cm}^2$$
$$l = \sqrt{h^2 + r^2}$$
$$= \sqrt{12^2 + 5^2}$$
$$= \sqrt{144 + 25}$$
$$= \sqrt{169} = 13 \text{ cm}$$

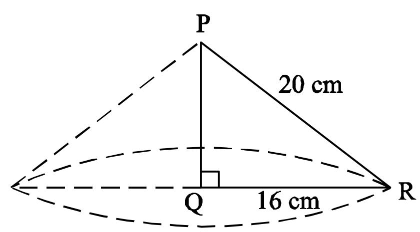
$$\text{Area of 1 cap} = \text{C.S.A of cone}$$
$$= \pi r l$$
$$= \frac{22}{7} \times 5 \times 13$$

$$\text{Number of caps} = \frac{\text{Total Area of paper}}{\text{Area of 1 cap}}$$
$$= \frac{5720}{\frac{22}{7} \times 5 \times 13}$$
$$= \frac{5720 \times 7}{22 \times 5 \times 13}$$
$$= 4 \times 7$$
$$= 28 \text{ caps}$$

---

4. A right angled triangle PQR where $\angle Q = 90°$ is rotated about QR and PQ. If $QR = 16$ cm and $PR = 20$ cm, compare the curved surface areas of the right circular cones so formed by the triangle.

Right angled triangle



In ΔPQR

$$PQ = \sqrt{PR^2 - QR^2}$$
$$= \sqrt{20^2 - 16^2}$$
$$= \sqrt{200 - 256}$$
$$= \sqrt{144}$$
$$\boxed{PQ = 12 \text{ cm}}$$

Case (i)

ΔPQR is rotated about QR

Here $h = QR = 16$ cm

$$r = PQ = 12 \text{ cm}$$
$$l = PR = 20 \text{ cm}$$
$$\therefore \text{C.S.A}_1 = \pi r l$$
$$= \pi \times 12 \times 20$$
$$= 240 \pi \text{ cm}^2$$

Case (ii)

ΔPQR is rotated about PQ

$$r = QR = 16 \text{ cm}$$
$$h = PQ = 12 \text{ cm}$$
$$l = PR = 20 \text{ cm}$$
$$\text{C.S.A}_2 = \pi r l$$
$$= \pi \times 16 \times 20$$
$$= 320\pi \text{ cm}^2$$

$\therefore$ C.S.A cone which is formed by rotating side PQ is larger.

---

7. The ratio of the radii of two right circular cones of same height is 1:3. Find the ratio of their curved surface area when the height of each cone is 3 times the radius of the smaller cone.

Cone

$$r_1 : r_2 = 1:3$$

Let

$$r_1 = x$$

$$r_2 = 3x$$

Given

$$h_1 = 3r_1 = 3x$$

$$h_2 = 3r_1 = 3x$$

$$l_1 = \sqrt{r_1^2 + h_1^2} = \sqrt{(x)^2 + (3x)^2} = \sqrt{x^2 + 9x^2} = \sqrt{10x^2}$$

$$l_1 = x\sqrt{10}$$

$$l_2 = \sqrt{r_2^2 + h_2^2} = \sqrt{(3x)^2 + (3x)^2} = \sqrt{9x^2 + 9x^2} = \sqrt{18x^2} = 3\sqrt{2}\,x$$

$$\therefore \text{C.S.A}_1 = \pi r_1 l_1 = \pi (x)(x)\sqrt{10} = \sqrt{10}\,\pi x^2$$

$$\text{C.S.A}_2 = \pi r_2 l_2 = \pi (3x)(3\sqrt{2}\,x) = 9\sqrt{2}\,\pi x^2$$

$$\frac{\text{C.S.A}_1}{\text{C.S.A}_2} = \frac{\sqrt{10}\,\pi x^2}{9\sqrt{2}\,\pi x^2} = \frac{\sqrt{5}\sqrt{2}}{9\sqrt{2}} = \frac{\sqrt{5}}{9}$$

$$\boxed{\textbf{C.S.A}_1 : \textbf{C.S.A}_2 = \sqrt{5} : 9}$$

Example 7.7

From a solid cylinder whose height is 2.4 cm and diameter 1.4 cm, a conical cavity of the same height and base is hollowed out (Fig.) Find the total surface area of the remaining solid.

Given

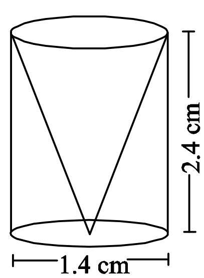
Cylinder

diameter $= 1.4$ cm

$$r = \frac{1.4}{2}$$

$$r = 0.7 \text{ cm}$$

$$h = 2.4 \text{ cm}$$



$$\therefore l = \sqrt{h^2 + r^2} = \sqrt{(2.4)^2 + (0.7)^2} = \sqrt{5.76 + 0.49} = \sqrt{6.25}$$

$$l = 2.5 \text{ cm}$$

Area of the remaining solid

$= \text{C.S.A of cylinder} + \text{C.S.A of cone} + \text{base area}$

$$= 2\pi r h + \pi r l + \pi r^2$$

$$= \pi r (2h + l + r)$$

$$= \frac{22}{7} \times 0.7 \times [(2 \times 2.4) + 2.5 + 0.7]$$

$$= 22 \times 0.1 \times [4.8 + 2.5 + 0.7]$$

$$= 2.2 \times 8$$

$$= 17.6 \text{ m}^2$$

Type III: Problems based on sphere, hemisphere, Hollow hemisphere

Example 7.8, 7.9, 8, 7.10, 7.11, 7.12, Q.No.9

Example 7.8

Find the diameter of a sphere whose surface area is 154 m².

Surface area $= 154 \text{ m}^2$

$$4\pi r^2 = 154$$

$$4 \times \frac{22}{7} \times r^2 = 154$$

$$r^2 = \frac{154 \times 7}{4 \times 22}$$

$$r^2 = \frac{7 \times 7}{4}$$

$$r = \frac{7}{2}$$

$\therefore$ diameter $= 2r$

$= 2\left(\frac{7}{2}\right)$

$= 7$ m

Example 7.9

The radius of a spherical balloon increases from 12 cm to 16 cm as air being pumped into it. Find the ratio of the surface area of the balloons in the two cases.

Spherical balloon

$$r_1 : r_2 = 16:12$$

$$\frac{r_1}{r_2} = \frac{16}{12}$$

$$= \frac{4}{3}$$

$$\frac{\text{C.S.A}_1}{\text{C.S.A}_2} = \frac{4\pi\, r_1^2}{4\pi\, r_2^2}$$

$$= \frac{r_1^2}{r_2^2}$$

$$= \left(\frac{r_1}{r_2}\right)^2$$

$$= \left(\frac{3}{4}\right)^2$$

$$= \frac{9}{16}$$

$\therefore$ ratio of C.S.A of balloons is 9:16

8. The radius of a sphere increases by 25%. Find the percentage increase in its surface area.

Sphere

Let radius $=$ 'r'

$\therefore$ Surface area $= 4\pi\, r^2$

New radius $= r + 25\%\, r$

$$= r + \frac{25}{100} r$$

$$= r + \frac{1}{4} r$$

$$R = \frac{5r}{4}$$

$\therefore$ Surface Area $= 4\pi\, R^2$

$$= 4\pi\left(\frac{5r}{4}\right)^2$$

$$= 4\pi \times \frac{25r^2}{16}$$

$$= \frac{25\pi\, r^2}{4}$$

$\therefore$ Increase S.A $= \frac{25\pi\, r^2}{4} - 4\pi\, r^2$

$$= \frac{9\pi\, r^2}{4}$$

$\therefore$ % Increase in S.A

$$= \frac{\text{Increase S.A}}{\text{Original S.A}} \times 100$$

$$= \frac{\frac{9\pi\, r^2}{4}}{4\pi\, r^2} \times 100$$

$$= \frac{9}{16} \times 100$$

$$= \frac{225}{4}$$

$$= 56.25\%$$

Example 7.10

If the base area of a hemispherical solid is 1386 sq.metres, then find its total surface area?

Hemisphere

Given

base area $= 1386$ sq.m

$\pi\, r^2 = 1386$ sq.m

$\therefore$ T.S.A $= 3\pi r^2$

$= 3 \times 1386$

$= 4158 \text{ m}^2$

Example 7.11

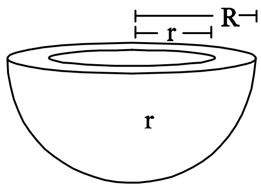
The internal and external radii of a hollow hemispherical steel are 3 m and 5 m respectively. Find the T.S.A and C.S.A of the shell.

Hollow hemispherical shell

Given

$r = 3$ cm

$R = 5$ cm



- C.S.A $= 2\pi (R^2 + r^2)$

$= 2 \times \frac{22}{7} (25 + 9)$

$= 2 \times \frac{22}{7} \times 34$

$= \frac{1496}{7}$

$= 213.71 \text{ m}^2$

- T.S.A $= \pi (3R^2 + r^2)$

$= \frac{22}{7} [3 (25) + 9]$

$= \frac{22}{7} [75 + 9]$

$= \frac{22}{7} \times 84$

$= 22 \times 12$

$= 264 \text{ m}^2$

9. The internal and external diameters of a hollow hemispherical vessel are 20 cm and 28 cm respectively. Find the cost of paint the vessel all over at Rs. 0.14 per cm^2.

Hollow hemispherical vessel

internal diameter $= 20$ cm

$r = 10$ cm

external diameter $= 28$ cm

$R = 14$ cm

T.S.A of the vessel $= \pi (3R^2 + r^2)$

$= \pi [3 (14)^2 + 10^2]$

$= \pi [3 (196) + 100]$

$= \frac{22}{7} [588 + 100]$

$= \frac{22}{7} \times 688$

$= \frac{15136}{7}$

$= 2162.28 \text{ cm}^2$

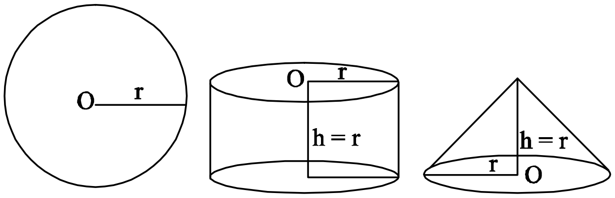
Cost of printing per $\text{cm}^2 =$ Rs. 0.14

$\therefore$ Total cost $= 2162.28 \times 0.14$

$=$ Rs. 302.72

Example 7.12

A sphere, a cylinder and a cone (Fig.) are of the same radius, where as cone and cylinder are of same height. Find the ratio of their curved surface areas.



Sphere: cylinder: cone

Here $l = \sqrt{r^2 + h^2}$

$= \sqrt{r^2 + r^2}$

$= \sqrt{2r^2}$

$= \sqrt{2} \cdot r$

Required ratio of C.S.A

= C.S.A of sphere: C.S.A of cylinder: C.S.A of cone

$= 4\pi r^2 : 2\pi r h : \pi r l$

$= 4\pi r^2 : 2\pi r (r) : \pi r (\sqrt{2}) r$

$= 4\pi r^2 : 2\pi r^2 : \sqrt{2} \pi r^2$

$= 4 : 2 : \sqrt{2}$

Divide by $\sqrt{2}$

$$= \frac{4}{\sqrt{2}} : \frac{2}{\sqrt{2}} : \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{2\sqrt{2}\sqrt{2}}{\sqrt{2}} : \frac{\sqrt{2}\cdot\sqrt{2}}{\sqrt{2}} : 1$$

$$= 2\sqrt{2} : \sqrt{2} : 1$$

Type IV: Problems based on Frustum of a cone

Example 7.13, 7.14, 10

Example 7.13

The slant height of a frustrum of a cone is 5 cm and the radii of its ends are 4 cm and 1 cm. Find its curved surface area.

Frustum of a cone

$l = 5$ cm

$R = 4$ cm

$r = 1$ cm

C.S.A $= \pi (R + r) l$

$$= \frac{22}{7}(4 + 1) \times 5$$

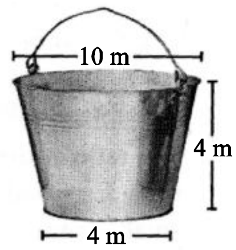
$$= \frac{22}{7} \times 5 \times 5$$

$$= \frac{550}{7}$$

C.S.A $= 78.57 \text{ cm}^2$

Example 7.14

An industrial metallic bucket is in the shape of the frustum of a right circular cone whose top and bottom diameters are 10 m and 4 m whose height is 4 m. Find the curved and total surface area of the bucket.



Metallic bucket

diameter of the top = 10 m

$\boxed{\therefore\ R = 5 \text{ m}}$

diameter of the bottom = 4 m

$\boxed{r = 2 \text{ m}}$

$h = 4$ m

Now $l = \sqrt{h^2 + (R - r)^2}$

$= \sqrt{4^2 + (5 - 2)^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

$l = 5$ m

- C.S.A $= \pi (R + r) l$

$$= \frac{22}{7}(5 + 2) \times 5$$

$$= \frac{22}{7} \times 7 \times 5$$

$$= 110 \text{ m}^2$$

- T.S.A $= \pi (R + r) l + \pi R^2 + \pi r^2$

$$= \pi [(R + r) l + R^2 + r^2]$$

$$= \frac{22}{7}[7(5) + 5^2 + 2^2]$$

$$= \frac{22}{7}[35 + 25 + 4]$$

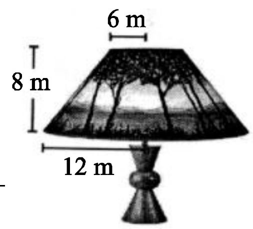
$$= \frac{22 \times 64}{7}$$

$$= \frac{1408}{7}$$

$$= 201.14 \text{ m}^2$$

10. The frustrum shaped outer portion of the table lamp to be painted including the top part. Find the total cost of painting the lamp if the cost of painting 1 sq.cm. is Rs. 2.

Table lamp (Frustrum shape)



Given $R = 12$ cm

$r = 6$ cm

$h = 8$ cm

$$l = \sqrt{h^2 + (R-r)^2}$$

$$= \sqrt{8^2 + (12-6)^2}$$

$$= \sqrt{8^2 + 6^2}$$

$$= \sqrt{64 + 36}$$

$$= \sqrt{100}$$

$$= 10 \text{ cm}$$

$\therefore$ C.S.A $= \pi (R + r)\, l$

$= \pi (12 + 6)\, 10$

$= \pi \times 18 \times 10$

$= 180\, \pi \text{ cm}^2$

Area of top $= \pi r^2$

$= \pi \times 6 \times 6$

$= 36\, \pi \text{ cm}^2$

$\therefore$ Total Area to be painted

$= 180\pi + 36\pi$

$= 216\pi \text{ cm}^2$

$= 216 \times 3.14$

$= 678.24 \text{ cm}^2$

Cost of painting per $\text{cm}^2 =$ Rs. 2

Total cost $= 678.24 \times 2$

$=$ Rs.1356.48

Exercise 7.2

KEY POINTS

Volume

1. Volume of a solid right circular cylinder = base Area $\times$ height

$$\boxed{V = \pi r^2 h \text{ cu.u}}$$

2. Volume of a hollow cylinder

= Volume of outer cylinder

$-$ Volume of inner cylinder

$$= \pi R^2 h - \pi r^2 h$$

$$\boxed{V = \pi h (R^2 - r^2) \text{ cu.u}}$$

3. Volume of a right circular

Cone $= \frac{1}{3} \times$ Volume of cylinder

$$\boxed{V = \frac{1}{3} \pi r^2 h \text{ cu.u}}$$

4. Volume of sphere

$$\boxed{V = \frac{4}{3} \pi r^3 \text{ cu.u}}$$

5. Volume of hollow sphere

$$V = \frac{4}{3} \pi (R^3 - r^3) \text{ cu.u}$$

6. Volume of hemisphere

$$V = \frac{2}{3} \pi r^3 \text{ cu.u}$$

7. Volume of hollow hemisphere

$$V = \frac{2}{3} \pi (R^3 - r^3) \text{ cu.u}$$

8. Volume of frustrum of a cone

$$V = \frac{\pi h}{3} (R^2 + Rr + r^2) \text{ cu.u}$$

Type I: Problems based on cylinder and Hollow cylinder

Q.No. 1, 2, Example 7.15, 7.16, 7.18, 7.17

1. A 14 m deep well with inner diameter 10 m is dug and the earth taken out is evenly spread all around the well to form an embankment of width 5 m. Find the height of the embankment.

Cylindrical well

diameter = 10 m

$r = 5$ m

$h = 14$ m

Volume of earth taken from the well

$$= \pi r^2 h$$

$$= \frac{22}{7} \times 5 \times 5 \times 14$$

$$= 22 \times 5 \times 5 \times 2$$

$$= 1100 \text{ cm}^3$$

Embankment (Hollow cylinder)

$$W = 5 \text{ m}$$

$$\therefore r = 5 \text{ m}$$

$$R = r + W$$

$$= 5 + 5$$

$$= 10 \text{ m}$$

$$h_1 = ?$$

Volume of earth taken from the well = Volume of the embankment

$$1100 = \pi (R^2 - r^2)\, h_1$$

$$1100 = \frac{22}{7}(100 - 25)\, h_1$$

$$1100 = \frac{22}{7} \times 75 \times h_1$$

$$h_1 = \frac{1100 \times 7}{22 \times 25}$$

$$= \frac{14}{3}$$

$$h = 4.67 \text{ cm}$$

$\therefore$ Height of the embankment is 4.67 cm.

2. A cylindrical glass with diameter 20 cm has water to a height of 9 cm. A small cylindrical metal of radius 5 cm and height 4 cm is immersed it completely. Calculate the raise of the water in the glass?

Cylindrical glass	**Cylindrical metal**
diameter = 20 cm	$r = 5$ cm
$R = 10$ cm	$h = 4$ cm
$H = 9$ cm	

Total volume of water in the glass

$$= \pi R^2 H + \pi r^2 h$$

$$= \pi [R^2 H + r^2 h]$$

$$= \pi [100 (9) + 25 (4)]$$

$$= \pi [900 + 100]$$

$$= 1000\pi \text{ cm}^3$$

Now,

$$\pi r_1^2 h_1 = 1000\, \pi \text{ cm}^3$$

$$\pi \times 10 \times 10 \times h_1 = 1000\pi$$

New water level $h_1 = 10$ cm

$\therefore$ Raise of the water level $= h_1 - H$

$$= 10 - 9$$

$$= 1 \text{ cm}$$

Example 7.15

Find the volume of a cylinder whose height is 2 m and whose base area is 250 m^2.

Cylinder

$h = 2$ m

base area $(\pi r^2) = 250 \text{ m}^2$

Volume $= \pi r^2 h$

$$= 250 \times 2$$

$$= 500 \text{ m}^3$$

Example 7.16

The volume of a cylindrical water tank is 1.078×10^6 litres. If the diameter of the tank is 7 m, find its height.

Cylindrical tank

Given

$$\text{Volume} = 1.078 \times 10^6$$

$$V = 1078000 \text{ litre}$$

$$= \frac{1078000}{1000} \text{ m}^3$$

$$V = 1078 \text{ m}^3$$

Given

$$\text{diameter} = 7 \text{ m}$$

$$r = \frac{7}{2} \text{ m}$$

$$\therefore \pi r^2 h = 1078$$

$$\frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times h = 1078$$

$$\frac{11 \times 7}{2} \times h = 1078$$

$$h = \frac{1078 \times 2}{11 \times 7}$$

$$h = 14 \times 2$$

$$h = 28 \text{ m}$$

$\therefore$ Height of the cylindrical tank is 28 m

Example 7.17

Find the volume of the iron used to make a hollow cylinder of height 9 cm and whose internal and external radii are 21 cm and 28 cm respectively.

Hollow cylinder

$$h = 9 \text{ cm}$$

$$R = 28 \text{ cm}$$

$$r = 21 \text{ cm}$$

$$\text{Volume} = \pi (R^2 - r^2)\, h \text{ cu.u}$$

$$= \frac{22}{7}(28^2 - 21^2) \times 9$$

$$= \frac{22}{7}(784 - 441) \times 9$$

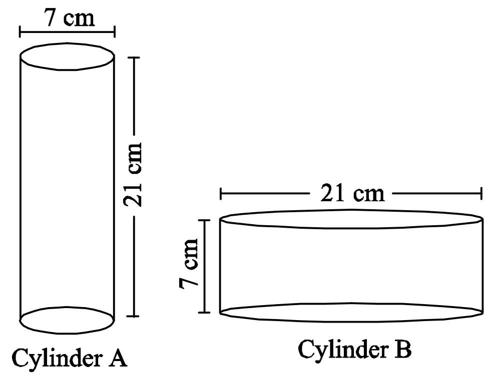
$$= \frac{22}{7} \times 343 \times 9$$

$$= 22 \times 49 \times 9$$

$$V = 9702 \text{ cm}^3$$

Example 7.18

For the cylinders A and B (Fig.),
(i) find out the cylinder whose volume is greater.
(ii) verify whether the cylinder with greater volume has greater total surface area.
(iii) find the ratios of the volumes of the cylinders A and B.



Cylinder A	**Cylinder B**
diameter $= 7$ cm	diameter $= 21$ cm
$r = \frac{7}{2}$ cm	$r = \frac{21}{2}$ cm
$h = 21$ cm	$h = 7$ cm

(i) Volume of cylinder $A = \pi r^2 h$

$$= \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 21$$

$$= \frac{11 \times 7 \times 21}{2}$$

$$= \frac{1617}{2}$$

$$= 808.5 \text{ cm}^3$$

Volume of cylinder $B = \pi r^2 h$

$$= \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times 7$$

$$= \frac{11 \times 21 \times 21}{2}$$

$$= \frac{4851}{2}$$

$$= 2425.5 \text{ cm}^3$$

$\therefore$ Volume of cylinder B is greater than volume of cylinder A

(ii) T.S.A of cylinder $A = 2\pi r (h + r)$

$$= 2 \times \frac{22}{7} \times \frac{7}{2}\left(21 + \frac{7}{2}\right)$$

$$= 2 \times \frac{22}{7} \times \frac{7}{2} \times \frac{49}{2}$$

$$= 11 \times 49$$

$$= 539 \text{ cm}^2$$

T.S.A of cylinder $B = 2\pi r (h + r)$

$$= 2 \times \frac{22}{7} \times \frac{21}{2}\left(7 + \frac{21}{2}\right)$$

$$= 2 \times \frac{22}{7} \times \frac{21}{2} \times \frac{35}{2}$$

$$= 11 \times 21 \times 5$$

$$= 1155 \text{ cm}^2$$

$\therefore$ Cylinder B with greater volume has a greater surface Area.

(iii) $\dfrac{\text{Volume of cylinder } A}{\text{Volume of cylinder } B} = \dfrac{808.5}{2425.5}$

$$= \frac{8085}{24255}$$

$$= \frac{1}{3}$$

ratio of volume of cylinder A and B is 1:3.

Type II: Problems based on volume of cone

Q.No. 3, 5, Example 7.19, 7.20, 6, 4

3. If the circumference of a conical wooden piece is 484 cm then find its volume when its height is 105 cm.

Conical wooden piece

$h = 105$ cm

base circumference $= 484$ cm

$$2\pi r = 484$$

$$2 \times \frac{22}{7} \times r = 484$$

$$r = \frac{484 \times 7}{2 \times 22}$$

$$r = 11 \times 7$$

$$\boxed{r = 77 \text{ cm}}$$

$\therefore$ Volume $= \frac{1}{3}\pi r^2 h$ cu.u

$$V = \frac{1}{3} \times \frac{22}{7} \times 77 \times 77 \times 105$$

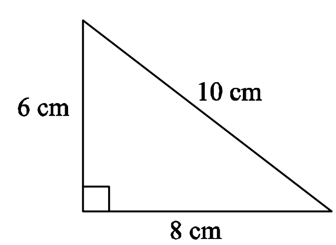
$$= 22 \times 11 \times 77 \times 35$$

$$V = 6{,}52{,}190 \text{ cm}^3$$

5. A right angled triangle whose sides are 6 cm, 8 cm and 10 cm is revolved about the sides containing the right angle in two ways. Find the difference in volumes of the two solids so formed.

Right angled triangle

Sides 6 cm, 8 cm, 10 cm

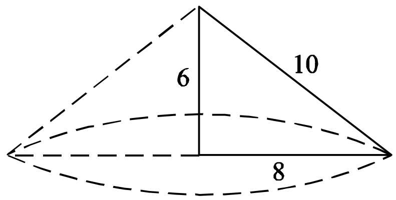


Case (i)

When triangle rotate about the side 6 cm as axis.

We get $h = 6$ cm

$r = 8$ cm



$$\text{Volume } = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\times\pi\times 8\times 8\times 6$$

$$= 128\pi \text{ cm}^3$$

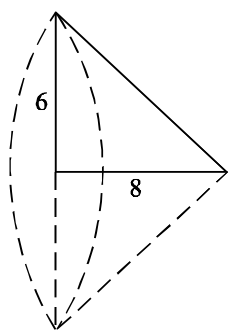
Case (ii)

When triangle rotate about the side 8 cm as axis.

We get

$h = 8$ cm

$r = 6$ cm



$$\text{Volume } = \frac{1}{3}\pi r^2 h$$

$$= \frac{\pi}{3}\times 6\times 6\times 8$$

$$= 96\pi \text{ cm}^3$$

$$\text{Difference in volumes } = 128\pi - 96\pi$$

$$= 32\pi \text{ cm}^3$$

$$= 32\times 3.14$$

$$= 100.48 \text{ cm}^3$$

Example 7.19

The volume of a solid right circular cone is 11088 cm³. If its height is 24 cm then find the radius of the cone.

Cone

Volume $= 11088 \text{ cm}^3$

$h = 24$ cm

$r = ?$

$$\frac{1}{3}\pi r^2 h = 11088 \text{ cm}^2$$

$$\frac{1}{3}\times\frac{22}{7}\times r^2\times 24 = 11088$$

$$r^2 = \frac{11088\times 3\times 7}{22\times 24}$$

$$r^2 = 21\times 3\times 7$$

$$r^2 = 3\times 7\times 3\times 7$$

$$r = 3\times 7$$

$$\boxed{r = 21 \text{ cm}}$$

Example 7.20

The ratio of the volumes of two cones is 2:3. Find the ratio of their radii if the height of second cone is double the height of the first.

Cone 1: Cone 2

radii, $r_1 : r_2$

height of ratio, Given $h_2 = 2h_1$

$$\therefore h_1 : h_2 = h_1 : 2h_1$$

$$\frac{\text{Volume of cone } I}{\text{Volume of cone } II} = \frac{2}{3}$$

$$\frac{\frac{1}{3}\pi r_1^2\times h_1}{\frac{1}{3}\pi r_2^2\times h_2} = \frac{2}{3}$$

$$\frac{r_1^2\times h_1}{r_2^2\times 2h_1} = \frac{2}{3}$$

$$\frac{r_1^2}{r_2^2} = \frac{4}{3}$$

$$\frac{r_1}{r_2} = \sqrt{\frac{4}{3}}$$

$$\boxed{r_1 : r_2 = 2 : \sqrt{3}}$$

---

6. The volumes of two cones of same base radius are 3600 cm^3 and 5040 cm^3. Find the ratio of heights.

Volume of two cones with same base radii 3600 cm^3 and 5040 cm^3

$$\frac{\text{Volume of cone} - I}{\text{Volume of cone} - II} = \frac{3600}{5040}$$

$$\frac{\frac{1}{3}\pi r^2 \times h_1}{\frac{1}{3}\pi r^2 \times h_2} = \frac{3600}{5040}$$

$$\frac{h_1}{h_2} = \frac{5}{7}$$

$$\boxed{h_1 : h_2 = 5 : 7}$$

---

4. A conical container is fully filled with petrol. The radius is 10 m and the height is 15 m. If the container can release the petrol through its bottom at the rate of 25 cu. meter per minute, in how many minutes the container will be emptied. Round off your answer to the nearest minute.

Conical container

$$r = 10 \text{ m}$$

$$h = 15 \text{ m}$$

Volume of petrol in the container

$$= \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3} \times \frac{22}{7} \times 10 \times 10 \times 15 \text{ m}^3$$

Given

Releasing petrol per minute $= 25 \text{ m}^3$

$$\therefore \text{ Time taken } = \frac{\frac{1}{3} \times \frac{22}{7} \times 10 \times 10 \times 15}{25}$$

$$= \frac{1}{3} \times \frac{22}{7} \times \frac{10}{25} \times 10 \times 15$$

$$= \frac{1}{3} \times \frac{22}{7} \times 60$$

$$= \frac{22}{7} \times 20$$

$$= \frac{440}{7}$$

$$= 63 \text{ minutes (approx)}$$

Type III: Problems based on sphere
Q.No. 7, 8, 9, Example 7.21, 7.22

7. If the ratio of radii of two spheres is 4:7, find the ratio of their volumes.

Sphere 1: Sphere 2

$$r_1 : r_2 = 4 : 7$$

$$V_1 : V_2 = \frac{4}{3}\pi r_1^3 : \frac{4}{3}\pi r_2^3$$

$$= (4)^3 : (7)^3$$

$$= 64 : 343$$

---

8. A solid sphere and a solid hemisphere have equal total surface area. Prove that the ratio of their volume is $3\sqrt{3} : 4$.

Given

T.S.A of sphere $=$ T.S.A of hemisphere

$$4\pi r_1^2 = 3\pi r_2^2$$

$$4r_1^2 = 3r_2^2$$

$$\frac{r_1^2}{r_2^2} = \frac{3}{4}$$

$$\left(\frac{r_1}{r_2}\right)^2 = \frac{3}{4}$$

$$\frac{r_1}{r_2} = \frac{\sqrt{3}}{2}$$

$$\boxed{r_1 : r_2 = \sqrt{3} : 2}$$

$$\therefore V_1 : V_2 = \frac{4}{3}\pi r_1^3 = \frac{4}{3}\pi r_2^3$$

$$= (\sqrt{3})^3 : (2)^3$$

$$V_1 : V_2 = 3\sqrt{3} : 8$$

---

9. The outer and the inner surface areas of a spherical copper shell are 576π cm^2 and 324π cm^2 respectively. Find the volume of the material required to make the shell.

Spherical copper shell

outer surface area $= 576\pi \text{ cm}^2$

inner surface area $= 324\pi \text{ cm}^2$

- $4\pi R^2 = 576\pi \text{ cm}^2$

$$R^2 = \frac{576}{4}$$

$$R^2 = 144$$

$$\boxed{R = 12 \text{ cm}}$$

- $4\pi r^2 = 324\pi \text{ cm}^2$

$$r^2 = \frac{324}{4}$$

$$r^2 = 81$$

$$\boxed{r = 9 \text{ cm}}$$

Volume $= \frac{4}{3}\pi (R^3 - r^3)$ cu.u

$$= \frac{4}{3} \times \frac{22}{7}(12^3 - 9^3)$$

$$= \frac{4}{3} \times \frac{22}{7}(1728 - 729)$$

$$= \frac{4}{3} \times \frac{22}{7} \times 999$$

$$= \frac{4 \times 22 \times 333}{7}$$

$$= \frac{29304}{7}$$

$$= 4186.285$$

$$V = 4186.29 \text{ cu.cm}$$

$\therefore$ The volume of the material needed $=$ 4186.29 cu.cm

---

Example 7.21

The volume of a solid hemisphere is 29106 cm^3. Another hemisphere whose volume is two-third of the above is carved out. Find the radius of the new hemisphere.

Volume of hemisphere $= 29106 \text{ cm}^3$

Volume of new hemisphere

$$= \frac{2}{3}\text{(Volume of Given hemisphere)}$$

$$= \frac{2}{3} \times 29106$$

$$= 2 \times 9702$$

$$V = 19404 \text{ cm}^3$$

$$\frac{2}{3}\pi r^3 = 19404$$

$$\frac{2}{3} \times \frac{22}{7} \times r^3 = 19404$$

$$r^3 = \frac{19404 \times 3 \times 7}{2 \times 22}$$

$$= 441 \times 3 \times 7$$

$$r^3 = 21 \times 21 \times 21$$

$$\boxed{r = 21 \text{ cm}}$$

Example 7.22

***Calculate the weight of a hollow brass sphere if the inner diameter is 14 cm and thickness is 1 mm, and whose density is* 17.3 g/cm³.**

inner diameter $= 14$ cm

$r = 7$ cm

thickness $w = 1$ mm

$= \frac{1}{10}$ cm

$w = 0.1$ cm

$\therefore R = r + w$

$= 7 + 0.1$

$= 7.1$ cm

Volume of hollow

Sphere $= \frac{4}{3}\pi (R^3 - r^3)$ cu.cm

$= \frac{4}{3} \times \frac{22}{7}\left[(7.1)^3 - (7)^3 \right]$

$= \frac{4}{3} \times \frac{22}{7} (357.91 - 343)$

$= \frac{4}{3} \times \frac{22}{7} \times 14.91$

$= \frac{4}{3} \times 22 \times 2.13$

$= 4 \times 22 \times 0.71$

$= 62.48$ cm^3

Given

Weight of brass in 1 cm^3 = 17.3 gm

Total weight $= 17.3 \times 62.48$

$= 1080.90$ gm

Type IV: Problems based on Frustrum of cone

Example 7.23, 10

Example 7.23

If the radii of the circular ends of a frustum which is 45 cm high are 28 cm and 7 cm, find the volume of the frustrum.

Given

$h = 45$ cm

$R = 28$ cm

$r = 7$ cm

Volume $= \frac{1}{3}\pi h (R^2 + Rr + r^2)$ cu.u

$= \frac{1}{3} \times \frac{22}{7} \times 45\, [28^2 + 28 + 7 + 7^2]$

$= \frac{22}{7} \times 15\, [784 + 196 + 49]$

$= \frac{22}{7} \times 15 \times 1029$

$= 22 \times 15 \times 147$

Volume of the frustrum of cone $= 48{,}510$ cm^3

10. A container open at the top is in the form of a frustrum of a cone of height 16 cm with radii of its lower and upper ends are 8 cm and 20 cm respectively. Find the cost of milk which can completely fill a container at the rate of Rs. 40 per litre.

Frustrum of cone

$h = 16$ cm

$R = 20$ cm

$r = 8$ cm

Volume $= \frac{1}{3}\pi h (R^2 + Rr + r^2)$ cu.u

$= \frac{1}{3} \times \frac{22}{7} \times 16\, (20^2 + 20 \times 8 + 8^2)$

$= \frac{1}{3} \times \frac{22}{7} \times 16\, (400 + 160 + 64)$

$$= \frac{1}{3} \times \frac{22}{7} \times 16 \times 624$$

$$= \frac{22}{7} \times 16 \times 208$$

$$= \frac{73216}{7}$$

$$= 10459.428 \text{ cm}^3$$

$$= \frac{10459.428}{1000}\, l$$

$$= 10.459$$

Capacity of milk = 10.46 litre

Cost of milk per litre = Rs. 40

Total cost $= 10.46 \times 40$

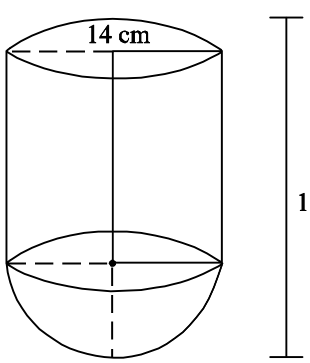
$= \text{Rs. } 418.40$

Exercise 7.3

Volume and Surface area of combined solids

Type I: Problems based on volume
Q.No. 1, 2, 3, 5, 4, Example 7.26

1. A vessel is in the form of hemispherical bowl mounted by a hollow cylinder. The diameter is 14 cm and the height of the vessel is 13 cm. Find the capacity of the vessel.



Cylinder

diameter = 14 cm

$r = 7$ cm

$h = 13 - 7$

$= 6$ cm

Hemisphere

$r = 7$ cm

Capacity of the vessel = Volume of cylindrical part + Volume of hemispherical part

$$= \pi r^2 h + \frac{2}{3}\pi r^3$$

$$= \pi r^2 \left(h + \frac{2r}{3} \right)$$

$$= \frac{22}{7} \times 7 \times 7 \left(6 + \frac{14}{3} \right)$$

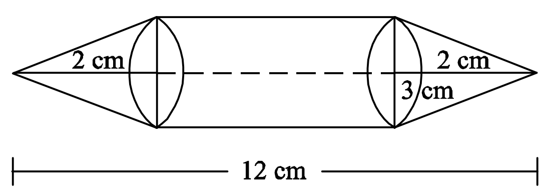
$$= \frac{22}{7} \times 7 \times 7 \times \frac{32}{3}$$

$$= \frac{4928}{3}$$

$V = 1642.67 \text{ cm}^3$

Capacity of the vessel is 1642.67 cm^3

2. Nathan, an engineering student was asked to make a model shaped like a cylinder with two cones attached at its two ends. The diameter of the model is 3 cm and its length is 12 cm. If each cone has a height of 2 cm, find the volume of the model that Nathan made.



Cone

$r = \frac{3}{2}$ cm

$h = 2$ cm

Cylinder

$r = \frac{3}{2}$ cm

$h = 12 - (2 + 2)$

$= 12 - 4$

$= 8$ cm

- Volume of cylindrical part $= \pi r^2 h$

$$= \pi \times \frac{3}{2} \times \frac{3}{2} \times 8$$

$$= 18\pi \text{ cm}^3$$

- Volume of 2 conical part $= 2 \times \frac{1}{3}\pi r^2 h$

$$= 2 \times \frac{1}{3} \times \pi \times \frac{3}{2} \times \frac{3}{2} \times 2$$

$$= 3\pi \text{ cm}^3$$

- Total volume $= 18\pi + 3\pi$

$= 21\pi$

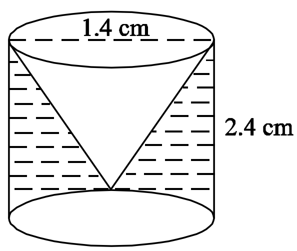
$= 21 \times \frac{22}{7}$

$= 3 \times 22$

$= 66 \text{ cm}^3$

Volume of the model is 66 cm^3

3. From a solid cylinder whose height is 2.4 cm and the diameter 1.4 cm, a cone of the same height and same diameter is carved out. Find the volume of the remaining solid to the nearest cm^3.



Cylinder	**Cone**
$r = \frac{1.4}{2}$	$r = 0.7$ cm
$= 0.7$ cm	$h = 2.4$ cm
$h = 2.4$ cm	

Volume of the remaining solid = Volume of cylinder − Volume of cone

$= \pi r^2 h - \frac{1}{3}\pi r^2 h$

$= \frac{2}{3}\pi r^2 h$

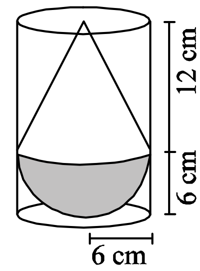
$= \frac{2}{3} \times \frac{22}{7} \times 0.7 \times 0.7 \times 2.4$

$= 2 \times 22 \times 0.1 \times 0.7 \times 0.8$

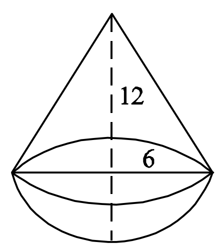
$= 2.464$

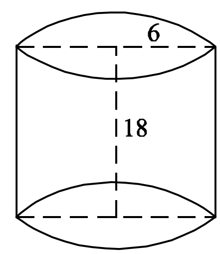
$= 2.46 \text{ cm}^3$

4. A solid consisting of a right circular cone of height 12 cm and radius 6 cm standing on a hemisphere of radius 6 cm is placed upright in a right circular cylinder full of water such that it touches the bottom. Find the volume of the water displaced out of the cylinder, if the radius of the cylinder is 6 cm and height is 18 cm.



Cone + hemisphere	**Cylinder**
$h = 12$ cm	$r = 6$ cm
$r = 6$ cm	$h = 18$ cm





- Volume of cone + hemisphere

$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$

$= \frac{1}{3}\pi r^2 (h + 2r)$

$= \frac{1}{3}\pi \times 6 \times 6\,(12 + 2\,(6))$

$= \pi \times 2 \times 6 \times 24$

$= 288\pi \text{ cm}^3$

Volume of water displaced in the cylinder

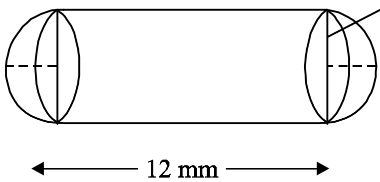
$= 288\pi \text{ cm}^3$

$= 288 \times \frac{22}{7}$

$= \frac{6336}{7}$

$= 905.14 \text{ cm}^3$

5. A capsule is in the shape of a cylinder with two hemisphere stuck to each of its ends. If the length of the entire capsule is 12 mm and the diameter of the capsule is 3 mm, how much medicine it can hold?



Cylinder

Diameter $= 3$ mm

$$r = \frac{3}{2}\text{ mm}$$

2 hemispherical part = 1 sphere

$$r = \frac{3}{2}\text{ mm}$$

$$h = 12 - \left(\frac{3}{2} + \frac{3}{2}\right)$$

$$= 12 - 3$$

$$h = 9\text{ mm}$$

$$\text{Total volume} = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$= \pi r^2\left(h + \frac{4}{3}r\right)$$

$$= \frac{22}{7} \times \frac{3}{2} \times \frac{3}{2}\left(9 + \frac{4}{3} \times \frac{3}{2}\right)$$

$$= \frac{22}{7} \times \frac{3}{2} \times \frac{3}{2}(9 + 2)$$

$$= \frac{22}{7} \times \frac{3}{2} \times \frac{3}{2} \times 11$$

$$= \frac{11 \times 9 \times 11}{14}$$

$$= \frac{1089}{14}$$

$$= 77.79\text{ mm}^3\text{ (app)}$$

$\therefore$ The capsule can hold 77.79 mm^3 of medicine.

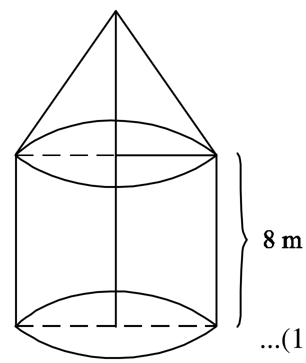
Example 7.26

Arul has to make arrangements for the accommodation of 150 persons for his family function. For this purpose, he plans to build a tent which is in the shape of cylinder surmounted by a cone. Each person occupies 4 sq.m of the space on ground and 40 cu.meter of air to breathe. What should be the height of the conical part of the tent if the height of cylindrical part is 8 m?

Tent

Given: Area of 1 person $= 4$ sq.m

Area of 150 person $= 150 \times 4$



$$\pi r^2 = 600\text{ m}^3$$

$$r^2 = 600 \times \frac{7}{22}$$

$$r^2 = 600 \times \frac{7}{22}$$

$$r^2 = \frac{300 \times 7}{11}$$

$$r^2 = \frac{2100}{11} \quad \text{...(1)}$$

Volume of air required for 1 person $= 40\text{ m}^3$

150 person $= 150 \times 40$

$= 6000\text{ m}^3$

Volume of tent $= 6000\text{ m}^3$

$$\pi r^2 h_1 + \frac{1}{3}\pi r^2 h_2 = 6000$$

$$\pi r^2\left(h_1 + \frac{h_2}{3}\right) = 6000$$

$$\frac{22}{7} \times \frac{2100}{11}\left(8 + \frac{h_2}{3}\right) = 6000$$

$$8 + \frac{h_2}{3} = \frac{6000 \times 7 \times 11}{22 \times 2100}$$

$$8 + \frac{h_2}{3} = 10$$

$$\frac{h_2}{3} = 10 - 8$$

$$h_2 = 2 \times 3$$

$$h_2 = 6\text{ cm}$$

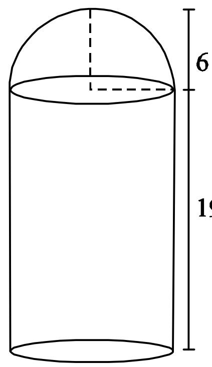
$\therefore$ Height of the conical tent is 6 cm.

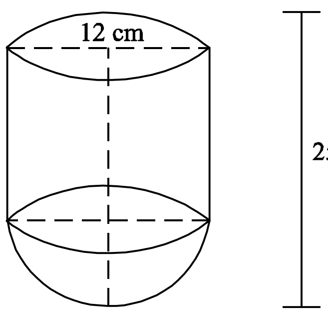
Type II: Problems based on surface area of combined solids

Example 7.24, 7.28, 6. 7.25, 7, 8, 7.27

Example 7.24

A toy is in the shape of a cylinder surmounted by a hemisphere. The height of the toy is 25 cm. Find the total surface area of the toy if its common diameter is 12 cm.





Cylinder	**Hemisphere**
diameter $= 12$ cm	$r = 6$ cm
$r = 6$ cm	
$h = 25 - 6$	
$= 19$ cm	

T.S.A of the toy = C.S.A of cylinder + C.S.A of hemisphere + base area of cylinder

$$= 2\pi r h + 2\pi r^2 + \pi r^2$$

$$= \pi r (2h + 2r + r)$$

$$= \pi r (2h + 3r)$$

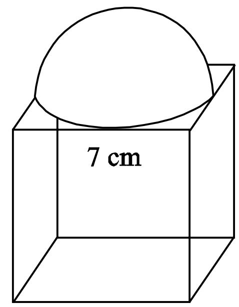
$$= \frac{22}{7} \times 6\ (38 + 18)$$

$$= \frac{22}{7} \times 6 \times 56$$

$$= 22 \times 6 \times 8$$

$$= 1056 \text{ cm}^2$$

6. As shown in figure a cubical block of side 7 cm is surmounted by a hemisphere. Find the surface area of the solid.



Cube	**Hemisphere**
Side $a = 7$ cm	diameter $= 7$ cm
	$r = \frac{7}{2}$ cm

T.S.A of the solid = T.S.A of cube + C.S.A of hemisphere − base area of hemisphere

$$= 6a^2 + 2\pi r^2 - \pi r^2$$

$$= 6a^2 + \pi r^2$$

$$= (6 \times 7 \times 7) + \left(\frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}\right)$$

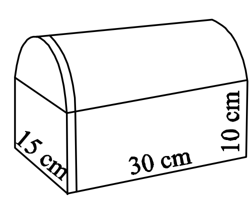
$$= 294 + \frac{77}{2}$$

$$= 294 + 38.5$$

$$= 332.5 \text{ cm}^2$$

Example 7.25

A jewel box (Fig.) is in the shape of a cuboid of dimensions 30 cm × 15 cm × 10 cm surmounted by a half part of a cylinder as shown in the figure. Find the volume and T.S.A of the box.



Cuboid	$\frac{1}{2}$ **cylinder**
$L = 30$ cm	$r = \frac{15}{2}$ cm
$B = 15$ cm	
$H = 10$ cm	$h = 30$ cm

$$= 6l^2 + \pi \frac{\pi l^2}{4}$$

$$= \frac{24l^2 + \pi l^2}{4}$$

$$= \frac{1}{4}(24 + \pi)\, l^2 \text{ sq. units}$$

Exercise 7.4

Conversion of solids from one shape to another with no change in volume

Type I: Find radius, height $V_1 = V_2$

Example 7.30, 1, 4, 6, 7, 3

Example 7.30

A cone of height 24 cm is made up of modeling day. A child reshapes it in the form of a cylinder of same radius as cone. Find the height of the cylinder.

Cone	Cylinder
$h_1 = 24$ cm	$h_2 = ?$

radius of cone = radius of cylinder

Volume of cylinder = Volume of cone

$$\pi r^2 h_2 = \frac{1}{3}\pi r^2 h_1$$

$$h_2 = \frac{1}{3} \times h_1$$

$$h_2 = \frac{1}{3} \times 24$$

$$h_2 = 8 \text{ cm}$$

$\therefore$ The height of cylinder is 8 cm.

---

1. An aluminium sphere of radius 12 cm is melted to make a cylinder of radius 8 cm. Find the height of the cylinder.

Sphere

$r_1 = 12$ cm

Cylinder

$r_2 = 8$ cm

$h = ?$

Volume of sphere = Volume of cylinder

$$\frac{4}{3}\pi r_1^3 = \pi r_2^2 h$$

$$\frac{4}{3} \times 12 \times 12 \times 12 = 8 \times 8 \times h$$

$$\frac{4 \times 4 \times 12 \times 12}{8 \times 8} = h$$

$$\boxed{36 \text{ cm} = h}$$

$\therefore$ Height of the cylinder is 36 cm

---

4. A solid right circular cone of diameter 14 cm and height 8 cm is melted to form a hollow sphere. If the external diameter of the sphere is 10 cm, find the internal diameter.

Cone

diameter $= 14$ cm

$r_1 = 7$ cm

$h_1 = 8$ cm

hollow sphere

external diameter $= 10$ cm

$R = 5$ cm

$r = ?$

Volume of cone = Volume of hollow sphere

$$\frac{1}{3}\pi r_1^2 h_1 = \frac{4}{3}\pi (R^3 - r^3)$$

$$\frac{1}{3} \times 7 \times 7 \times 8 = \frac{4}{3}(5^3 - r^3)$$

$$\frac{1}{3} \times 7 \times 7 \times 8 \times \frac{3}{4} = 125 - r^3$$

$$98 = 125 - r^3$$

$$r^3 = 125 - 98$$

$$r^3 = 27$$

$$\boxed{r = 3 \text{ cm}}$$

$\therefore$ internal diameter of the hollow sphere is $3 \times 2 = 6$ cm

---

6. The internal and external diameter of a hollow hemispherical shell are 6 cm and 10 cm respectively. If it is melted and recast into a solid cylinder of diameter 14 cm, then find the height of the cylinder.

Hollow hemi-spherical shell	**Solid cylinder**
$r = \frac{6}{2} = 3$ cm	$r_1 = \frac{14}{2} = 7$ cm
$R = \frac{10}{2} = 5$ cm	$h = ?$

Volume of hollow hemispherical shell = Volume of solid cylinder

$$\frac{2}{3}\pi (R^3 - r^3) = \pi r_1^2 h$$

$$\frac{2}{3}(5^3 - 3^3) = 7 \times 7 \times h$$

$$\frac{2}{3}(125 - 27) = 7 \times 7 \times h$$

$$\frac{2}{3} \times 98 = 7 \times 7 \times h$$

$$\frac{2 \times 98}{3 \times 7 \times 7} = h$$

$$\frac{4}{3} = h$$

$$\boxed{\therefore\ h = 1.33 \text{ cm}}$$

---

7. A solid sphere of radius 6 cm is melted into a hollow cylinder of uniform thickness. If the external radius of the base of the cylinder is 5 cm and its height is 32 cm, then find the thickness of the cylinder.

Solid sphere	**Hollow cylinder**
$r_1 = 6$ cm	$R = 5$ cm
	$h = 32$ cm
	$r = ?$
	Thickness $W = ?$

Volume of solid sphere = Volume of hollow cylinder

$$\frac{4}{3}\pi r_1^3 = \pi (R^2 - r^2) h$$

$$\frac{4}{3} \times 6 \times 6 \times 6 = (5^2 - r^2)\, 32$$

$$\frac{4 \times 2 \times 6 \times 6}{32} = 25 - r^2$$

$$9 = 25 - r^2$$

$$r^2 = 16$$

$$\boxed{r = 4 \text{ cm}}$$

$\therefore$ thickness $(w) = R - r$

$= 5 - 4$

$= 1$ cm

3. A conical flask is full of water. The flask has base radius *r* units and height *h* units, the water poured into a cylindrical flask of base radius *xr* units. Find the height of water in the cylindrical flask.

Conical flask	**Cylindrical flask**
radius $= r$	base radius $= xr$
height = h	height $(H) = ?$

Volume of water in conical flask = Volume of water in cylindrical flask

$$\frac{1}{3}\pi r^2 h = \pi r^2 H$$

$$\frac{1}{3} r^2 h = (xr)^2 \times H$$

$$\frac{1}{3} r^2 h = x^2 r^2 \times H$$

$$\frac{h}{3x^2} = H$$

$\therefore$ Height of water in the cylindrical flask is $\dfrac{h}{3x^2}$

Type II: $\dfrac{V_1}{V_2}$ based sums

Example 7.29, 7.31, 2, 5, 8

Example 7.29

A metallic sphere of radius 16 cm is melted and recast into small spheres each of radius 2 cm. How many small spheres can be obtained?

Sphere

$R = 16$ cm

Small sphere

$r = 2$ cm

Number of small spheres

$$= \frac{\text{Volume of bigger sphere}}{\text{Volume of 1 small sphere}}$$

$$= \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi r^3}$$

$$= \frac{R^3}{r^3}$$

$$= \frac{16 \times 16 \times 16}{2 \times 2 \times 2}$$

$$= 16 \times 16 \times 2$$

$$= 512 \text{ spheres}$$

2. Water is flowing at the rate of 15 km per hour through a pipe of diameter 14 cm into a rectangular tank which is 50 m long and 44 m wide. Find the time in which the level of water in the tanks will rise by 21 cm.

Cylindrical pipe

diameter $= 14$ cm

$$r = 7 \text{ cm} = \frac{7}{100} \text{ m}$$

Speed $(h) = 15$ km/hr

$$= 15 \times 1000$$

$$= 15000 \text{ m}$$

Rectangular tank

$$L = 50 \text{ m}$$

$$B = 44 \text{ m}$$

$$H = 21 \text{ cm}$$

$$= \frac{21}{100} \text{ m}$$

Time taken

$$= \frac{\text{Volume of water in Rectangular tank}}{\text{Volume of water flowing in pipe in 1 hour}}$$

$$= \frac{LBH}{\pi r^2 h}$$

$$= \frac{50 \times 44 \times \frac{21}{100}}{\frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 15000}$$

$$= 50 \times 44 \times \frac{21}{100} \times \frac{7}{22} \times \frac{100}{7} \times \frac{100}{7} \times \frac{1}{15000}$$

$$= 2 \text{ hours}$$

Example 7.31

A right circular cylindrical container of base radius 6 cm and height 15 cm is full of ice cream. The ice cream is to be filled in cones of height 9 cm and base radius 3 cm, having a hemispherical cap. Find the number of cones needed to empty the container.

Cylindrical container	Ice cream cup (cone + hemisphere)
$r = 6$ cm	$h = 9$ cm
$h = 15$ cm	$r = 3$ cm

Volume of ice cream in cylindrical container

$$= \pi r^2 h$$

$$V_1 = \pi \times 6 \times 6 \times 15 \quad ...(1)$$

Volume of ice cream in 1 cup = Volume of cone + Volume of hemisphere

$$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$$

$$= \frac{1}{3}\pi r^2 (h + 2r)$$

$$= \frac{1}{3}\pi \times 3 \times 3 \,(9 + 6)$$

$$V_2 = \pi \times 3 \times 15 \quad ...(2)$$

$\therefore$ Number of ice cream cups $= \dfrac{V_1}{V_2}$

$$= \frac{\pi \times 6 \times 6 \times 15}{\pi \times 3 \times 15}$$

$= 12$ cups

---

5. Seenu's house has an overhead tank in the shape of a cylinder. This is filled by pumping water from a sump (underground tank) which is in the shape of a cuboid. The sump has dimensions 2 m × 1.5 m × 1 m. The overhead tank has its radius of 60 cm and height 105 cm. Find the volume of the water left in the sump after the overhead tank has been completely filled with water from the sump which has been full, initially.

Sump (cuboid)	Overhead tank (cylinder)
$L = 2$ m $= 200$ cm	$r = 60$ cm
$B = 1.5$ m $= 150$ cm	$h = 105$ cm
$H = 1$ m $= 100$ cm	

- Volume of water in the sump $= LBH$

$$= 200 \times 150 \times 100$$

$$= 30{,}00{,}000 \text{ cm}^3$$

- Volume of water in the overhead tank $\pi r^2 h$

$$= \frac{22}{7} \times 60 \times 60 \times 105$$

$$= 22 \times 60 \times 60 \times 15$$

$$= 11{,}88{,}000 \text{ cm}^3$$

$\therefore$ The volume of water left in the sump

$$= 30{,}00{,}000 - 11{,}88{,}000$$

$$= 18{,}12{,}000 \text{ cm}^3$$

---

8. A hemispherical bowl is filled to the brim with juice. The juice is poured into a cylindrical vessel whose radius is 50% more than its height. If the diameter is same for both the bowl and the cylinder then find the percentage of juice that can be transferred from the bowl into the cylindrical vessel.

Hemisphere	Cylinder
Radius $= r$	Radius $= r$
	$= h + \frac{1}{2}h$
	$r = \frac{3}{2}h$

- Volume of hemisphere $= \dfrac{2}{3}\pi r^3$

$$= \frac{2}{3}\pi\left(\frac{3}{2}h\right)^3$$

$$= \frac{2}{3}\times\pi\times\frac{27}{8}\times h^3$$

$$= \frac{9}{4}\pi h^3$$

- Volume of cylinder $= \pi r^2 h$

$$= \pi\left(\frac{3}{2}h\right)^2 \times h$$

$$= \pi\times\frac{9}{4}h^2\times h$$

$$= \frac{9}{4}\pi h^3$$

Here volume of hemisphere = Volume of cylinder

$\therefore$ % of juice that can be transferred to the cylindrical vessel = 100%

Exercise 7.6

Multiple choice questions

1. The curved surface area of a right circular cone of height 15 cm and base diameter 16 cm is

1. 6π cm^2 2. 68π cm^2
3. 120π cm^2 4. 136π cm^2

Cone

$$h = 15 \text{ cm}$$

$$r = 8 \text{ cm}$$

$$l = \sqrt{h^2 + r^2}$$

$$= \sqrt{15^2 + 8^2}$$

$$= \sqrt{225 + 64}$$

$$= \sqrt{289}$$

$$= 17 \text{ cm}$$

C.S.A $= \pi r l$

$= \pi \times 8 \times 17$

$= 136\pi$ cm^2

Ans. (4) 136π cm^2

2. If two solid hemispheres of same base radius r units are joined together along their bases, then curved surface area of this new solid is

1. $4\pi r^2$ sq. units 2. $6\pi r^2$ sq. units
3. $3\pi r^2$ sq. units 4. $8\pi r^2$ sq. units

When we join two hemispheres with same base radius we get a sphere

$\therefore$ Surface area of sphere $= 4\pi r^2$

Ans. (1) $4\pi r^2$ sq. u

3. The height of a right circular cone whose radius is 5 cm and slant height is 13 cm will be

1. 12 cm 2. 10 cm
3. 13 cm 4. 5 cm

Cone

$$r = 5 \text{ cm}$$

$$l = 13 \text{ cm}$$

$$h = \sqrt{l^2 - r^2}$$

$$= \sqrt{13^2 - 5^2}$$

$$= \sqrt{169 - 25}$$

$$= \sqrt{144}$$

$$= 12$$

Ans. (1) 12 cm

4. If the radius of the base of a right circular cylinder is halved keeping the same height, then the ratio of the volume of the cylinder thus obtained to the volume of original cylinder is

1. 1:2 2. 1:4
3. 1:6 4. 1:8

radius of cylinder $= r$

radius of New cylinder $R = \frac{r}{2}$

$$\therefore \frac{\text{Volume of New cylinder}}{\text{Volume of original cylinder}}$$

$$= \frac{\pi R^2 h}{\pi r^2 h}$$

$$= \frac{\left(\frac{r}{2}\right)}{r^2}$$

$$= \frac{r^2}{4} \times \frac{1}{r^2}$$

$$= \frac{1}{4}$$

Ans. (2) 1:4

---

5. The total surface area of a cylinder whose radius is $\frac{1}{3}$ of its height is

1. $\frac{9\pi h^2}{8}$ sq. units
2. $24\pi h^2$ sq. units
3. $\frac{8\pi h^2}{9}$ sq. units
4. $\frac{56\pi h^2}{9}$ sq. units

Cylinder

$$r = \frac{1}{3} h$$

$$\text{T.S.A} = 2\pi r (h + r)$$

$$= 2\pi \left(\frac{1}{3} h\right)\left(h + \frac{1}{3} h\right)$$

$$= 2\pi \times \frac{h}{3} \times \frac{4h}{3}$$

$$= \frac{8\pi h^2}{9}$$

Ans. (3) $\frac{8\pi h^2}{9}$ sq. u

---

6. In a hollow cylinder, the sum of the external and internal radii is 14 cm and the width is 4 cm. If its height is 20 cm, the volume of the material in it is

1. $5000\pi \text{ cm}^3$
2. $11200\pi \text{ cm}^3$
3. $56\pi \text{ cm}^3$
4. $3600\pi \text{ cm}^3$

Hollow cylinder

$$R + r = 14 \text{ cm}$$

$$W = 4 \text{ cm} \rightarrow (R - r)$$

$$h = 20 \text{ cm}$$

$$\text{Volume} = \pi (R^2 - r^2) h$$

$$= \pi (R + r)(R - r) h$$

$$= \pi \times 14 \times 4 \times 20$$

$$= 1120\pi \text{ cm}^3$$

Ans. (2) $1120\pi \text{ cm}^3$

---

7. If the radius of the base of a cone is tripled and the height is doubled then the volume is

1. made 6 times
2. made 18 times
3. made 12 times
4. unchanged

Cone

$$r = 3r$$

$$h = 2h$$

$$\text{Volume} = \frac{1}{3} \pi r^2 h$$

$$= \frac{1}{3} \pi (3r)^2 (2h)$$

$$= \frac{1}{3} \pi \times 9 \times r^2 \times 2h$$

$$= 18 \left(\frac{1}{3} \pi r^2 h\right)$$

$$= 18 \text{ Volume of cone}$$

Ans. (2) made 18 times

---

8. The total surface area of a hemi-sphere is how much times the square of its radius.

1. π
2. 4π
3. 3π
4. 2π

$$\text{T.S.A of hemisphere} = 3\pi r^2$$

$$= 3\pi \times \text{square of radius}$$

Ans. (3) 3π

9. A solid sphere of radius x cm is melted and cast into a shape of a solid cone of same radius. The height of the cone is

1. $3x$ cm 2. x cm
3. $4x$ cm 4. $2x$ cm

Sphere

radius = 'x' cm

Cone

radius = 'x' cm

height = ?

Volume of sphere = Volume of cone

$$\frac{4}{3}\pi x^3 = \frac{1}{3}\pi x^2 h$$

$$4x^3 = x^2 h$$

$$\frac{4x^3}{x^2} = h$$

$$4x = h$$

Ans. (3) $4x$ cm

---

10. A frustum of a right circular cone is of height 10 cm with radii of its ends as 8 cm and 20 cm. Then, the volume of the frustrum is

1. 3328π cm^3 2. 3228π cm^3
3. 3240π cm^3 4. 3340π cm^3

Frustrum of cone

$h = 16$ cm

$r = 8$ cm

$R = 20$ cm

$$\text{Volume} = \frac{1}{3}\pi h(R^2 + Rr + r^2) \text{ cu.u}$$

$$= \frac{1}{3}\times\pi\times 16\left[20^2 + 20\,(8) + 8^2\right]$$

$$= \frac{1}{3}\times\pi\times 16\,[400 + 160 + 64]$$

$$= \pi\times\frac{16}{3}\times 624$$

$$= \pi\times 16\times 208$$

$$V = 3328\pi \text{ cm}^3$$

Ans. (1) 3328π cm^3

---

11. A shuttle cock used for playing badminton has the shape of the combination of

1. a cylinder and a sphere
2. a hemisphere and a cone
3. a sphere and a cone
4. frustrum of a cone and a hemisphere

Ans. (4) frustrum of a cone and a hemisphere

12. A spherical ball of radius r_1 units is melted to make 8 new identical balls each of radius r_2 units. Then r_1:r_2 is

1. 2:1 2. 1:2
3. 4:1 4. 1:4

Volume of bigger sphere = Volume of 8 small sphere

$$\frac{4}{3}\pi r_1^3 = 8\times\frac{4}{3}\pi r_2^3$$

$$r_1^3 = 8r_2^3$$

$$\frac{r_1^3}{r_2^3} = \frac{8}{1}$$

$$\left(\frac{r_1}{r_2}\right)^3 = \left(\frac{2}{1}\right)^3$$

$$r_1 : r_2 = 2:1$$

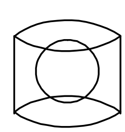
Ans. (1) 2:1

---

13. The volume (in cm^3) of the greatest sphere that can be cut off from a cylindrical log of wood of base radius 1 cm and height 5 cm is

1. $\frac{4}{3}\pi$ 2. $\frac{10}{3}\pi$
3. 5π 4. $\frac{20}{3}\pi$

Cylinder



radius = 1 cm

height = 5 cm

Volume of greatest sphere

Cut from cylinder with $r = 1$ cm

$$= \frac{4}{3}\pi r^3$$

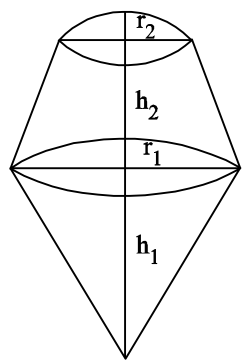
$$= \frac{4}{3}\pi (1)^3$$

$$= \frac{4}{3}\pi$$

Ans. (1) $\frac{4}{3}\pi$

14. The height and radius of the cone of which the frustum is a part are h_1 units and r_1 units respectively. Height of the frustum is h_2 units and radius of the smaller base is r_2 units. If $h_2 : h_1 = 1:2$ then $r_2 : r_1$ is

1. 1:3 2. 1:2
3. 2:1 4. 3:1



Given

$h_2 : h_1 = 1:2$

$$\frac{h_2}{h_1} = \frac{1}{2}$$

$\Rightarrow r_2 : r_1 = 1:2$

Ans. (2) 1:2

15. The ratio of the volumes of a cylinder, a cone and a sphere, if each has the same diameter and same height is

1. 1:2:3 2. 2:1:3
3. 1:3:2 4. 3:1:2

Cylinder : Cone : Sphere

$\pi r^2 h : \frac{1}{3}\pi r^2 h : \frac{4}{3}\pi r^3$

$\Rightarrow \pi r^2 h : \frac{1}{3}\pi r^2 h : \frac{4}{3}\pi r^2 \frac{h}{2}$

$1 : \frac{1}{3} : \frac{2}{3}$

$3 : 1 : 2$

Here in sphere $[2r = h]$ $r = \frac{h}{2}$ **Ans.** (4) 3:1:2

Exercise 7

1. The barrel of a fountain-pen cylindrical in shape, is 7 cm long and 5 mm in diameter. A full barrel of ink in the pen will be used for writing 330 words on an average. How many words can be written using a bottle of ink containing one fifth of a litre?

Fountain pen (cylinder)

$$h = 7 \text{ cm}$$

$$r = \frac{5}{2} \text{ mm}$$

$$= \frac{5}{2 \times 10}$$

$$r = \frac{1}{4} \text{ cm}$$

Volume $= \pi r^2 h$

$$= \frac{22}{7} \times \frac{1}{4} \times \frac{1}{4} \times 7$$

$$= \frac{11}{8} \text{ cm}^3$$

Given

$\frac{11}{8}$ cm^3 = 330 words

$\frac{1}{5}$ of litre $= \frac{1}{5} \times 1000$ cm^3 = 'x' words

200 cm^3 = x words

$$x = \frac{200}{\frac{11}{8}} \times 330$$

$$= 200 \times 330 \times \frac{8}{11}$$

$$= 200 \times 30 \times 8$$

$= 48000$ words

2. A hemi-spherical tank of radius 1.75 m is full of water. It is connected with a pipe which empties the tank at the rate of 7 litre per second. How much time will it take to empty the tank completely?

Hemispherical tank

$r = 1.75$ m

$r = 175$ cm

Volume of water flowing by the pipe

$= 7$ litre/sec

$= 7000 \text{ cm}^3$

$$\text{Time taken} = \frac{\text{Volume of water in tank}}{7000}$$

$$= \frac{\frac{2}{3}\pi r^3}{7000}$$

$$= \frac{2}{3} \times \frac{22}{7} \times 175 \times 175 \times \frac{1}{7000}$$

$$= \frac{11 \times 5 \times 175}{6}$$

$$= \frac{9625}{6}$$

$= 1604.17$ seconds

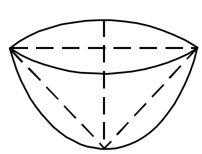
$$= \frac{1604.17}{60} \text{ minutes}$$

$= 26.74$

$= 27$ minutes (approx)

---

3. Find the maximum volume of a cone that can be carved out of a solid hemisphere of radius r units.



Hemisphere

radius $= r$ units

Cone

radius $= r$

height $= r$

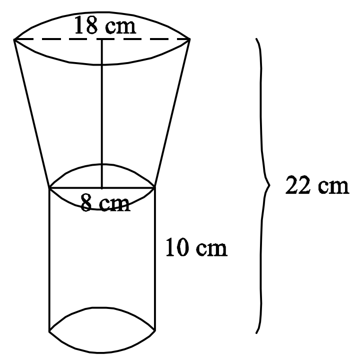
$$\therefore \text{ Maximum volume } = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2 (r)$$

$$= \frac{\pi r^3}{3} \text{ cu.u}$$

---

4. An oil funnel of tin sheet consists of a cylindrical portion 10 cm long attached to a frustum of a cone. If the total height is 22 cm, the diameter of the cylindrical portion be 8 cm and the diameter of the top of the funnel be 18 cm, then find the area of the tin sheet required to make the funnel.



Cylinder

diameter $= 8$ cm

$r = 4$ cm

$h = 10$ cm

$\therefore$ C.S.A $= 2\pi r h$

$= 2\pi \times 4 \times 10$

$= 80\pi \text{ cm}^2$

Frustum of cone

$h = 22 - 10$

$= 12$ cm

$$R = \frac{18}{2} = 9 \text{ cm}$$

$$r = \frac{8}{2} = 4 \text{ cm}$$

$$l = \sqrt{(R-r)^2 + h^2}$$

$$= \sqrt{(9-4)^2 + (12)^2}$$

$$= \sqrt{5^2 + 12^2}$$

$$l = \sqrt{25 + 144}$$

$$l = \sqrt{169}$$

$$\boxed{l = 13 \text{ cm}}$$

$$\text{C.S.A} = \pi (R + r) l$$

$$= \pi (9 + 4)(13)$$

$$= \pi \times 13 \times 13$$

$$= 169\pi \text{ cm}^2$$

$\therefore$ T.S.A of the funnel $= 80\pi + 169\pi$

$$= 249\pi \text{ cm}^2$$

$$= 249 \times \frac{22}{7}$$

$$= \frac{5478}{7}$$

$$= 782.57 \text{ cm}^2$$

---

5. Find the number of coins, 1.5 cm in diameter and 2 mm thick, to be melted to form a right circular cylinder of height 10 cm and diameter 4.5 cm.

Cylinder

$$h = 10 \text{ cm}$$

diameter $= 4.5$ cm

$$r = \frac{4.5}{2} \text{ cm}$$

Coin (cylinder)

diameter $= 1.5$ cm

$$r_1 = \frac{1.5}{2} \text{ cm}$$

thickness $(h_1) = 2$ mm

$$= \frac{2}{10} \text{ mm}$$

$$\text{Number of coins} = \frac{\text{Volume of solid cylinder}}{\text{Volume of 1 coin}}$$

$$= \frac{\pi r^2 h}{\pi r_1^2 h_1}$$

$$= \frac{\frac{4.5}{2} \times \frac{4.5}{2} \times 10}{\frac{1.5}{2} \times \frac{1.5}{2} \times \frac{2}{10}}$$

$$= \frac{4.5}{2} \times \frac{4.5}{2} \times 10 \times \frac{2}{1.5} \times \frac{2}{1.5} \times \frac{10}{2}$$

$$= 450 \text{ coins}$$

---

6. A hollow metallic cylinder whose external radius is 4.3 cm and internal radius is 1.1 cm and whole length is 4 cm is melted and recast into a solid cylinder of 12 cm long. Find the diameter of solid cylinder.

Hollow metallic cylinder

$$R = 4.3 \text{ cm}$$

$$r = 1.1 \text{ cm}$$

$$h = 4 \text{ cm}$$

Solid cylinder

$$h_1 = 12 \text{ cm}$$

$$r_1 = ?$$

Volume of hollow metallic cylinder = Volume of solid cylinder

$$\pi (R^2 - r^2) h = \pi r_1^2 h_1$$

$$[(4.3)^2 - (1.1)^2] 4 = r_1^2 \times 12$$

$$(4.3 + 1.1)(4.3 - 1.1) 4 = r_1^2 \times 12$$

$$\frac{5.4 \times 3.2 \times 4}{12} = r_1^2$$

$$1.8 \times 3.2 = r_1^2$$

$$5.76 = r_1^2$$

$$\boxed{r_1 = 2.4 \text{ cm}}$$

$\therefore$ diameter of solid cylinder $= 2r_1$

$$= 2(2.4)$$

$$= 4.8 \text{ cm}$$

7. The slant height of a frustrum of a cone is 4 m and the perimeter of circular ends are 18 m and 16 m. Find the cost of painting its curved surface area at Rs. 100 per sq.m.

Frustrum of cone

$l = 4$ m

$2\pi R = 18$ m and $2\pi r = 16$ m

$R = \frac{18}{2\pi}$	$r = \frac{16}{2\pi}$
$R = \frac{9}{\pi} m$	$r = \frac{8}{\pi} m$

$$\text{C.S.A} = \pi (R + r) l$$

$$= \pi \left(\frac{9}{\pi} + \frac{8}{\pi} \right) 4$$

$$= \pi \times \frac{17}{\pi} \times 4$$

$$= 68 \text{ m}^2$$

Cost of painting per m^2 = Rs. 100

$\therefore$ Total cost $= 68 \times 100$

$= \text{Rs.} 6800$

8. A hemi-spherical hollow bowl has material of volume $\frac{436\pi}{3}$ cubic cm. Its external diameter is 14 cm. Find its thickness.

Hollow hemispherical bowl

$$\text{Volume} = \frac{436\pi}{3} \text{ cm}^3$$

$$R = \frac{14}{2}$$

$$R = 7 \text{ cm}$$

$$\frac{2}{3} \pi (R^3 - r^3) = \frac{436\pi}{3}$$

$$2 (7^3 - r^3) = 436$$

$$343 - r^3 = \frac{436}{2}$$

$$343 - r^3 = 218$$

$$343 - 218 = r^3$$

$$125 = r^3$$

$$r^3 = 5^3$$

$$r = 5 \text{ cm}$$

$\therefore$ thickness $W = R - r$

$= 7 - 5$

$W = 2$ cm

9. The volume of a cone is $1005 \frac{5}{7}$ cu.cm. The area of its base is $201 \frac{1}{7}$ sq.cm. Find the slant height of the cone.

Cone

$$\text{Volume} = 1005 \frac{5}{7} \text{ cm}^3$$

$$= \frac{7040}{7} \text{ cm}^3$$

$$\text{Area of base } (\pi r^2) = 201 \frac{1}{7} \text{ cm}^2$$

$$= \frac{1408}{7} \text{ cm}^2$$

$$V = \frac{1}{3} \pi r^2 h$$

$$\frac{7040}{7} = \frac{1}{3} \times \frac{1408}{7} \times h$$

$$\frac{7040 \times 3}{1408} = h$$

$$\frac{21120}{1408} = h$$

$$\boxed{\mathbf{15 \text{ cm} = h}}$$

- $\pi r^2 = \frac{1408}{7} \text{ cm}^2$

$$r^2 = \frac{1408}{7} \times \frac{7}{22}$$

$$r^2 = 64$$

$$\boxed{\mathbf{r = 8 \text{ cm}}}$$

- $l = \sqrt{h^2 + r^2}$

$= \sqrt{15^2 + 8^2}$

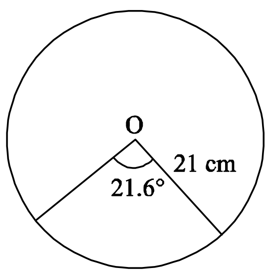
$= \sqrt{225 + 64}$

$= \sqrt{289}$

$$\boxed{l = 17 \text{ cm}}$$

---

10. A metallic sheet in the form of a sector of a circle of radius 21 cm has central angle of 216°. The sector is made into a cone by bringing the bounding radii together. Find the volume of the cone formed.



When a sector of a circle folded to form a cone, then we get

- Radius of sector = slant height of cone
- Area of sector = C.S.A of cone
- Length of arc = base perimeter

We have

$R = 21$ cm

$\therefore l = 21$ cm

Length of arc = base perimeter

$$\frac{\phi}{360} \times 2\pi R = 2\pi r$$

$$\frac{216}{360} \times 21 = r$$

$$\frac{4536}{360} = r$$

$$\boxed{12.6 \text{ cm} = r}$$

$h = \sqrt{l^2 - r^2}$

$= \sqrt{21^2 - 12.6^2}$

$= \sqrt{441 - 158.76}$

$= \sqrt{282.24}$

$$\boxed{h = 16.8 \text{ cm}}$$

$$\begin{array}{r|l}
 & 16.8 \\ \hline
1 & 282.24 \\
 & 1 \\ \hline
26 & 182 \\
 & 156 \\ \hline
328 & 2624 \\
 & 2624 \\ \hline
 & 0 \\ \hline
\end{array}$$

$\therefore$ Volume of the cone

$= \frac{1}{3} \pi r^2 h$

$= \frac{1}{3} \times \frac{22}{7} \times 12.6 \times 12.6 \times 16.8$

$= 22 \times 4.3 \times 1.8 \times 16.8$

$= 2860.70 \text{ cm}^3$

CHAPTER 8

STATISTICS AND PROBABILITY

Exercise 8.1

KEY POINTS

I. Measures of central tendency

1. Arithmetic mean
2. Median
3. Mode

II. Measures of Dispersion

1. Range
2. Mean deviation
3. Quartile deviation
4. Standard deviation
5. Variance
6. Co-efficient of variation

1. Range

- Range = Largest value − Smallest value

$$\boxed{R = L - S}$$

- Co-efficient of Range $= \dfrac{L-S}{L+S}$

2. Variance

$$\text{Variance } \sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$$

3. Standard Deviation

The positive square root of variance is called standard deviation.

Calculation of S.D for ungrouped data

(i) Direct method

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

(ii) Mean method

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma (x-\overline{x})^2}{n}}$$

If $d = x - \overline{x}$, then $\sigma = \sqrt{\dfrac{\Sigma d^2}{n}}$

(iii) Assumed mean method

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} \quad \text{where } d = x - A$$

(iv) Step deviation method

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2} \times C$$

Calculation of standard deviation for grouped data

(i) Mean method

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma f (x-\overline{x})^2}{N}}$$

let $d = x - \overline{x}$ then

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma fd^2}{N}} \quad \text{where } N = \Sigma f$$

(ii) Assumed mean method

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2}$$

Calculation of standard deviation for continuous frequency distribution

(i) Mean method

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{N}}$$

(ii) Short cut method (or) Step deviation method

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} \times C$$

where $d = \dfrac{x-A}{C}$

Note:

1. S.D of first 'n' natural numbers is

$$\sigma = \sqrt{\frac{n^2-1}{12}}$$

2. S.D will not change when we add or sub some fixed constant to all the values
3. When we multiply or divide each data by 'k' the S.D also get multiplied or divided by 'k'
4. S.D $= \sqrt{\text{variance}}$

   Variance $= (\text{S.D})^2$

Type I: Problems based on range and co-efficient of range

Q.No.1 (i) (ii), Example 8.1, 8.3, 2, 3, Example 8.2

1. Find the range and co-efficient of range of the following data:

(i) 63, 89, 98, 125, 79, 108, 117, 68

Largest value $= 125$

Smallest value $= 63$

$\therefore$ Range $= L - S$

$= 125 - 63$

$= 62$

$$\text{Co-efficient of Range} = \frac{L-S}{L+S} = \frac{125-63}{125+63} = \frac{31}{94} = 0.33$$

(ii) 43.5, 13.6, 18.9, 38.4, 61.4, 29.8

Largest value $(L) = 61.4$

Smallest value $(S) = 13.6$

$\therefore$ Range $= L - S$

$= 61.4 - 13.6$

$= 47.8$

$$\text{Co-efficient of Range} = \frac{L-S}{L+S} = \frac{61.4-13.6}{61.4+13.6} = \frac{47.8}{75} = 0.64$$

Example 8.1

Find the range and coefficient of range of the following data: 25, 67, 48, 53, 18, 39, 44.

Largest value $L = 67$; Smallest value $S = 18$

Range $R = L - S = 67 - 18 = 49$

Coefficient of range $= \dfrac{L-S}{L+S}$

Coefficient of range $= \dfrac{67-18}{67+18} = \dfrac{49}{85} = 0.576$

2. If the range and the smallest value of a set of data are 36.8 and 13.4 respectively, then find the largest value.

Given

Range = 36.8

Smallest value $= 13.4$

Largest value $= ?$

$$R = L - S$$

$$36.8 = L - 13.4$$

$$36.8 + 13.4 = L$$

$$\boxed{50.2 = L}$$

Example 8.3

The range of a set of data is 13.67 and the largest value is 70.08. Find the smallest value.

Range $R = 13.67$

Largest value $L = 70.08$

Range $R = L - S$

$13.67 = 70.08 - S$

$S = 70.08 - 13.67 = 56.41$

Therefore, the smallest value is 56.41.

3. Calculate the range of the following data.

Income	400-450	450-500	500-550	550-600	600-650
Number of workers	8	12	30	21	6

Largest value $= 650$

Smallest value $= 400$

$\therefore$ Range $= L - S$

$= 650 - 400$

$= 250$

Example 8.2

Find the range of the following distribution.

Age (in years)	16-18	18-20	20-22	22-24	24-26	26-28
Number of students	0	4	6	8	2	2

Here Largest value $L = 28$

Smallest value $S = 18$

Range $R = L - S$

$R = 28 - 18 = 10$ years

Type II: S.D ungrouped data

Q.No: 7, 6, 8, 9, Example 8.9, 8.10, 4, 5, Example 8.5, 8.6

7. Find the standard deviation of first 21 natural numbers.

S.D of first 'n' natural numbers

$$\sigma = \sqrt{\frac{n^2-1}{12}}$$

Here $n = 21$

$$S.D = \sqrt{\frac{21^2-1}{12}}$$

$$= \sqrt{\frac{441-1}{12}}$$

$$= \sqrt{\frac{440}{12}}$$

$$= \sqrt{36.67}$$

$$S.D \simeq 6.05$$

$$\begin{array}{r|l} & 6.05 \\ \hline 6 & 36.6666 \\ & 36 \\ \hline 1205 & \quad 6666 \\ & \quad 6025 \\ \hline & \quad\;\; 641 \end{array}$$

---

6. A wall clock strikes the bell once at 1 O' clock, 2 times at 2 O' clock, 3 times at 3 O' clock and so on. How many times will it strike in a particular day. Find the standard deviation of the number of strikes the bell make a day.

Given data

The bell strike in a particular day

$$= 2\,(1 + 2 + 3 + \ldots + 12)$$

$$= \frac{2 \times 12 \times 13}{2} \qquad \left[\Sigma n = \frac{n\,(n+1)}{2}\right]$$

$$= 12 \times 13$$

$$= 156 \text{ times}$$

$$\text{S.D} = 2 \times \sqrt{\frac{n^2 - 1}{12}}$$

$$= 2 \times \sqrt{\frac{12^2 - 1}{12}}$$

$$= 2 \times \sqrt{\frac{144 - 1}{12}}$$

$$= 2 \times \sqrt{\frac{143}{12}}$$

$$= 2 \times \sqrt{11.92}$$

$$= 2 \times 3.4$$

$$\text{S.D} \simeq 6.8$$

$$\begin{array}{r|l} & 3.4 \\ \hline 3 & 11.92 \\ & \;\;9 \\ \hline 64 & \;\;292 \\ & \;\;256 \\ \hline & \;\;\;\;36 \end{array}$$

---

8. If the standard deviation of a data is 4.5 and if each value of the data is decreased by 5, then find the new standard deviation.

We have

S.D will not change when we add or sub some fixed constant to all the values.

Here all the values decreased by 5.

$\therefore$ Now S.D = 4.5

---

9. If the standard deviation of a data is 3.6 and each value of the data is divided by 3, then find the new variance and new standard deviation.

We know that

When we multiply or divide each data by 'k' the S.D also get multiplied or divided by 'k'.

Here $k = 3.6$

$$\text{S.D} = 3$$

$$\therefore \text{ New S.D} = \frac{3.6}{3}$$

$$\text{S.D} = 1.2$$

$$\text{Variance} = (\text{S.D})^2$$

$$= 1.2 \times 1.2$$

$$= 1.44$$

Example 8.8

Find the standard deviation of the following data 7, 4, 8, 10, 11. Add 3 to all the values then find the standard deviation for the new values.

Arranging the values in ascending order we get, 4, 7, 8, 10, 11 and $n = 5$

x_i	x_i^2
4	16
7	49
8	64
10	100
11	121
$\Sigma x_i = 40$	$\Sigma x_i^2 = 350$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{350}{5} - \left(\frac{40}{5}\right)^2}$$

$$= \sqrt{70 - 64}$$

$$\sigma = \sqrt{6} = 2.45$$

When we add 3 to all the values, we get the new values are 7, 10, 11, 13, 14.

x_i	x_i^2
7	9
10	100
11	121
13	169
14	196
$\Sigma x_i = 55$	$\Sigma x_i^2 = 635$

Standard deviation

$$\Sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{635}{5} - \left(\frac{55}{5}\right)^2}$$

$$= \sqrt{127 - 121}$$

$$\sigma = \sqrt{6} = 2.45$$

From the above, we see that the standard deviation will not change when we add some fixed constant to all the values.

Example 8.9

Find the standard deviation of the data 2, 3, 5, 7, 8. Multiply each data by 4. Find the standard deviation of the new values.

Given, $n = 5$

x_i	x_i^2
2	49
3	9
5	25
7	49
8	64
$\Sigma x_i = 25$	$\Sigma x_i^2 = 151$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n^2}\right)^2}$$

$$\sigma = \sqrt{\frac{151}{5} - \left(\frac{25}{5}\right)^2}$$

$$= \sqrt{30.2 - 25}$$

$$= \sqrt{5.2} = 2.28$$

When the multiply each data by 4, we get the new values as 8, 12, 20, 28, 32.

x_i	x_i^2
8	64
12	144
20	400
28	784
32	1024
$\Sigma x_i = 100$	$\Sigma x_i^2 = 2416$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{2416}{5} - \left(\frac{100}{5}\right)^2}$$

$= \sqrt{483.2 - 400}$

$= \sqrt{83.2}$

$\sigma = \sqrt{16 \times 5.2}$

$= 4\sqrt{5.2} = 9.12$

From the above, we see that when we multiply each data by 4 the standard deviation also get multiplied by 4.

---

4. A teacher asked the students to complete 60 pages of a record note book. Eight students have completed only 32, 35, 37, 30, 33, 36, 35 and 37 pages. Find the standard deviation of the pages yet to be completed by them.

The pages yet to be completed by them are $60 - 32 = 28$, $60 - 35 = 25$, $60 - 7 = 23$, $60 - 30 = 30$, $60 - 33 = 27$, $60 - 26 = 24$, $60 - 35 = 25$, $60 - 37 = 23$

We have to find

$\therefore$ S.D of 28, 25, 23, 30, 27, 24, 25, 23

Note:

Here $\bar{x} = 25.625$ decimal so we Assumed mean method

x	$d = x - A$	d^2
23	$23 - 25 = -2$	4
23	$23 - 25 = -2$	4
24	$24 - 25 = -1$	1
25 A	$25 - 25 = 0$	0
25	$25 - 25 = 0$	0
27	$27 - 25 = 2$	4
28	$28 - 25 = 3$	9
30	$30 - 25 = 5$	25
	$\Sigma d = 5$	$\Sigma d^2 = 47$

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2}$$

$$= \sqrt{\frac{47}{8} - \left(\frac{5}{8}\right)^2}$$

$$= \sqrt{\frac{47}{8} - \frac{25}{64}}$$

$$= \sqrt{\frac{376 - 25}{64}}$$

$$= \frac{\sqrt{351}}{8}$$

$$= \frac{18.74}{8}$$

$$\sigma = 2.34$$

---

5. Find the variance and standard deviation of the wages of 9 workers given below Rs. 310, Rs. 290, Rs. 320, Rs. 280, Rs. 300, Rs. 290, Rs. 320, Rs. 310, Rs. 280.

x	$d = \frac{x - A}{C}$	d^2
280	-2	4
280	-2	4
290	-1	1
290	-1	1
300 A	0	0
310	1	1
310	1	1
320	2	4
320	2	4
	$\Sigma d = 0$	$\Sigma d^2 = 20$

$$\therefore \text{ S.D } (\sigma) = \sqrt{\frac{\Sigma d^2}{n} \times C}$$

$$= \sqrt{\frac{20}{9}} \times 100$$

$$= \sqrt{2.2222} \times 10$$

$$= \sqrt{2.22} \times 10$$

$$= 1.49 \times 10$$

$$\sigma = 14.9$$

$$\text{Variance } = (\text{S.D})^2$$

$$= [\sqrt{2.222} \times 10]^2$$

$$= 2.222 \times 100$$

$$= 222.2$$

Example 8.5

The amount of rainfall in a particular season for 6 days are given as 17.8 cm, 19.2 cm, 16.3 cm, 12.5 cm, 12.8 cm and 11.4 cm. Find its standard deviation.

Arranging the numbers in ascending order we get, 11.4, 12.5, 12.8, 16.3, 17.8, 19.2. [Number of observations $n = 6$]

$$\text{Mean } = \frac{11.4 + 12.5 + 12.8 + 16.3 + 17.8 + 19.2}{6}$$

$$= \frac{90}{6} = 15$$

x_i	$d_i = x_i - \bar{x} = x - 15$	d_i^2
11.4	− 3.6	12.96
12.5	− 2.5	6.25
12.8	− 2.2	4.84
16.3	1.3	1.69
17.8	2.8	7.84
19.2	4.2	17.64
		$\Sigma d_1^2 = 51.22$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$$

$$= \sqrt{\frac{51.22}{6}} = \sqrt{8.53}$$

Hence $\sigma = 2.9$

Example 8.6

The marks scored by 10 students in a class test are 25, 29, 30, 33, 35, 37, 38, 40, 44, 48. Find the standard deviation.

✍ **Solution:**

The mean of marks is 35.9 which is not an integer. Hence we take assumed mean, $A = 35$, $n = 10$.

x_1	$d_i = x_i - A$ $d_i = x - 35$	d_i^2
25	− 10	100
29	− 6	36
30	− 5	25
33	− 2	4
35	0	0
37	2	4
38	3	9
40	5	25
44	9	81
48	13	169
	$\Sigma d_i = 9$	$\Sigma d_i^2 = 453$

Example 8.10

Find the mean and variance of the first n natural numbers.

$1 + 2 + 3 + \ldots + n$

$$\text{mean } \bar{x} = \frac{\text{Sum of all observation}}{\text{Number of observation}}$$

$$= \frac{\Sigma x i}{n}$$

$$= \frac{1+2+3+\ldots+n}{n}$$

$$= \frac{n(n+1)}{2n}$$

$$\boxed{\textbf{Mean } \bar{x} = \frac{n+1}{2}}$$

Variance $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

$$= \frac{n(n+1)(2n+1)}{6 \times n} - \left[\frac{n(n+1)}{2n}\right]^2$$

$$= \frac{n(n+1)(2n+1)}{6 \times n} - \frac{n^2(n+1)^2}{4n^2}$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$= \frac{n+1}{2} \times \frac{n-1}{6}$$

$$\boxed{\textbf{Variance } \sigma^2 = \frac{n^2-1}{12}}$$

Note:

S.D $\sigma = \sqrt{\frac{n^2-1}{2}}$

Type III: S.D for grouped data

Q.No.10, Example 8.11, 8.12

10. The rainfall recorded in various place of five districts in a week are given below.

Rainfall (in mm)	45	50	55	60	65	70
Number of places	5	13	4	9	5	4

Find its standard deviation.

x	f	$d = \frac{x-A}{C}$	d^2	fd	fd^2
45	5	−2	4	−10	20
50	13	−1	1	−13	13
55 A	4	0	0	0	0
60	9	1	1	9	9
65	5	2	4	10	20
70	4	3	9	12	36
	$N = 40$			$\Sigma fd = 8$	$\Sigma fd^2 = 98$

$$\therefore \text{ S.D } (\sigma) = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} \times C$$

$$= \sqrt{\frac{98}{40} - \left(\frac{8}{40}\right)^2} \times 5$$

$$= \sqrt{\frac{98}{40} - \left(\frac{1}{5}\right)^2} \times 5$$

$$= \sqrt{\frac{98}{40} - \frac{1}{25}} \times 5$$

$$= \sqrt{\frac{490-8}{200}} \times 5$$

$$= \sqrt{\frac{482}{200}} \times 5$$

$$= \sqrt{2.41} \times 5$$

$$= 1.55 \times 5$$

$$\boxed{\textbf{S.D} \simeq \textbf{7.75}}$$

Example 8.11

48 students were asked to write the total number of hours per week they spent on watching television. With this information find the standard deviation of hours spent for watching television.

x	**6**	**7**	**8**	**9**	**10**	**11**	**12**
y	**3**	**6**	**9**	**13**	**8**	**5**	**4**

x_i	f_i	$x_i f_i$	$d_i = x_i - \overline{x}$	d_i^2	$f_i d_i^2$
6	3	18	– 3	9	27
7	6	42	– 2	4	24
8	9	72	– 1	1	9
9	13	117	0	0	0
10	8	80	1	1	8
11	5	55	2	4	20
12	4	48	3	9	36
	$N = 48$	$\Sigma x_i f_i = 432$	$\Sigma d_i = 0$		$\Sigma f_i d^2 = 124$

Mean $\overline{x} = \frac{\Sigma x_i f_i}{N} - \frac{432}{48} = 9$ (Since $N = \Sigma f_i$)

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}} = \sqrt{\frac{124}{48}} = \sqrt{2.58}$$

$$\sigma = 1.6$$

Example 8.12

The marks scored by the students in a slip test are given below.

x	**4**	**6**	**8**	**10**	**12**
f	**7**	**3**	**5**	**9**	**5**

Find the standard deviation of their marks.

Let the assumed mean, $A = 8$

x_i	f_i	$d_i = x_i - A$	$f_i d_i$	$f_i d_i^2$
4	7	– 4	– 28	112
6	3	– 2	– 6	12
8	5	0	0	0
10	9	2	18	36
12	5	4	20	80
	$N = 29$		$\Sigma f_i d_i = 4$	$\Sigma f_i d_i^2 = 240$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_1}{N}\right)^2}$$

$$= \sqrt{\frac{240}{29} - \left(\frac{4}{29}\right)^2} = \sqrt{\frac{240 \times 29 - 16}{29 \times 29}}$$

$$\sigma = \sqrt{\frac{6944}{29 \times 29}}\ ; \sigma = 2.87$$

Type IV: S.D for continuous frequency distribution

Q.No. 11, 12, 13, Example 8.13

11. In a study about viral fever, the number of people affected in a town were noted as

Age in years	**0-10**	**10-20**	**20-30**	**30-40**	**40-50**	**50-60**	**60-70**
Number of people affected	**3**	**5**	**16**	**18**	**12**	**7**	**4**

Find its standard deviation

Age in years	Mid value x	f	$d=\frac{x-A}{c}$	d^2	fd	fd^2
0-10	5	3	– 3	9	– 9	27
10-20	15	5	– 2	4	– 10	20
20-30	25	16	– 1	1	– 16	16
30-40	**35** A	18	0	0	0	0
40-50	45	12	1	1	12	12
50-60	55	7	2	4	14	28
60-70	65	4	3	9	12	36
		$N=65$			$\Sigma fd=3$	$\Sigma fd^2 = 139$

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} \times C$$

$$= \sqrt{\frac{139}{65} - \left(\frac{3}{65}\right)^2} \times 10$$

$$= \sqrt{\frac{139}{65} - \frac{9}{65^2}} \times 10$$

$$= \sqrt{\frac{9035-9}{65^2}} \times 10$$

$$= \frac{\sqrt{9026}}{65} \times 10$$

$$= \frac{95.005}{65} \times 10$$

$$= \frac{950.05}{65}$$

$$\text{S.D } (\sigma) = 14.6$$

12. The measurements of the diameters (in cms) of the plates prepared in a factory are given below. Find its standard deviation.

Diameter (in cm)	**21-24**	**25-28**	**29-32**	**33-36**	**37-40**	**41-44**
Number of plates	**15**	**18**	**20**	**16**	**8**	**7**

Dia meter in cm	Mid value x	f	$d=\frac{x-A}{C}$	d^2	fd	fd^2
21-24	22.5	15	– 2	4	– 30	60
25-28	26.5	18	– 1	1	– 18	18
29-32	**30.5** A	20	0	0	0	0
33-36	34.5	16	1	1	16	16
37-40	38.5	8	2	4	16	32
41-44	42.5	7	3	9	21	63
		$N=94$			$\Sigma fd=5$	$\Sigma fd^2 = 189$

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2} \times C$$

$$\sqrt{\frac{189}{94} - \left(\frac{5}{94}\right)^2} \times 4$$

$$= \sqrt{\frac{189}{94} - \frac{25}{94^2}} \times 4$$

$$= \sqrt{\frac{17766-25}{94^2}} \times 4$$

$$= \frac{\sqrt{17741}}{94} \times 4$$

$$= \frac{133.20}{94} \times 4$$

$$= 1.42 \times 4$$

$$\boxed{\sigma \simeq 5.68}$$

13. The time taken by 50 students to complete a 100 meter race are given below. Find its standard deviation.

Time taken (sec)	**8.5-9.5**	**9.5-10.5**	**10.5-11.5**	**11.5-12.5**	**12.5-13.5**
Number of students	**6**	**8**	**17**	**10**	**9**

Time taken (in sec)	**Mid value (x)**	f	$d = x - A$	d^2	fd	fd^2
8.5-9.5	9	6	− 2	4	− 12	24
9.5-10.5	10	8	− 1	1	− 8	8
10.5-11.5	**11 A**	17	0	0	0	0
11.5-12.5	12	10	1	1	10	10
12.5-13.5	13	9.2	2	4	18	36
		$N = 50$			$\Sigma fd = 8$	$\Sigma fd^2 = 78$

$$\text{S.D} = \sqrt{\frac{\Sigma fd^2}{N} - \left(\frac{\Sigma fd}{N}\right)^2}$$

$$= \sqrt{\frac{78}{50} - \left(\frac{8}{50}\right)^2}$$

$$= \sqrt{\frac{78}{50} - \frac{64}{50^2}}$$

$$= \sqrt{\frac{3900 - 64}{50^2}}$$

$$= \frac{\sqrt{3836}}{50}$$

$$= \frac{61.94}{50}$$

$$\boxed{\sigma \simeq 1.24}$$

Example 8.13

Marks of the students in a particular subject of a class are given below.

Marks	**0-10**	**10-20**	**20-30**	**30-40**	**40-50**	**50-60**	**60-70**
Number of students	**8**	**12**	**7**	**14**	**9**	**7**	**4**

Find its standard deviation.

Let the assumed mean, $A = 35$, $c = 10$

Marks	**Mid value (x_1)**	f_i	$d_i = x_i - A$	$d_i = \frac{x_i - A}{c}$	$f_i d_i$	$f_i d_i^2$
0-10	5	8	− 30	3	− 24	72
10-20	15	12	− 20	− 2	− 24	48
20-30	25	17	− 10	− 1	− 17	17
30-40	35	14	0	0	0	0
40-50	45	9	10	1	9	9
50-60	55	7	20	2	14	28
60-70	65	4	30	3	12	36
		$N = 71$			$\Sigma f_i d_i = -30$	$\Sigma f_i d_i^2 = 210$

Standard deviation $\sigma = c \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$

$$\sigma = 10 \times \sqrt{\frac{210}{71} - \left(-\frac{30}{71}\right)^2} = 10 \times \sqrt{\frac{210}{71} - \frac{900}{5041}}$$

$$= 10 \times \sqrt{2.779}\,;\ \sigma = 16.67$$

Type V: S.D and variance special sums
Q.No.14, Example 8.14, 15

Example 8.14

The mean and standard deviation of 15 observations are found to be 10 and 5 respectively. On rechecking it was found that one of the observation with value 8 was incorrect. Calculate the correct mean and standard deviation if the correct observation value was 23?

$n = 15, \bar{x} = 10, \sigma = 15;$

$\bar{x} = \frac{\Sigma x}{n}; \Sigma x = 15 \times 10 = 150$

Wrong observation value $= 8$,

Correct observation value $= 23$.

Correct total $= 150 - 8 + 23 = 165$

Correct mean $x = \frac{165}{15} = 11$

Standard deviation $\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$

Incorrect value of $\sigma = 5 = \sqrt{\frac{\Sigma x^2}{15} - (10)^2}$

$25 = \frac{\Sigma x^2}{15} - 100$ gives $\frac{\Sigma x^2}{15} = 125$

Incorrect value of $\Sigma x^2 = 1875$

Correct value of

$$\Sigma x^2 = 1875 - 8^2 + 23^2 = 2340$$

Correct standard deviation

$$\sigma = \sqrt{\frac{2340}{15} - (11)^2}$$

$\sigma = \sqrt{156 - 121} = \sqrt{35}$ $\sigma = 5.9$

---

14. For a group of 100 candidates the mean and standard deviation of their marks were found to be 60 and 15 respectively. Later on it was found that the scores 45 and 72 were wrongly entered as 40 and 27. Find the correct mean and standard deviation.

Given

$$n = 100$$

$$\bar{x} = 60$$

$$\sigma = 15,$$

$$\bar{x} = \frac{\Sigma x}{n}$$

$$60 = \frac{\Sigma x}{100}$$

$$\Sigma x = 6000$$

Wrong observation values $= 40, 27$

Correct observation values $= 45, 72$

$\therefore$ Correct total

$$\Sigma x = 6000 - (40 + 27) + (45 + 72)$$

$$= 6000 - 67 + 117$$

$$= 6050$$

Correct mean $\bar{x} = \frac{6050}{100}$

$$\boxed{\bar{x} = \mathbf{60.5}}$$

- Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$$

$$15 = \sqrt{\frac{\Sigma x^2}{100} - (60)^2}$$

$$15^2 = \frac{\Sigma x^2}{100} - (60)^2$$

$$225 = \frac{\Sigma x^2}{100} - 3600$$

$$225 + 3600 = \frac{\Sigma x^2}{100}$$

$$3825 \times 100 = \Sigma x^2$$

$$382500 = \Sigma x^2 \text{ (in correct)}$$

Correct $\Sigma x^2 = 382500 - 40^2 - 27^2 + 45^2 + 72^2$

$$= 382500 - 1600 - 729 + 2025 + 5184$$

$$= 387380$$

$\therefore$ Correct S.D $= \sqrt{\frac{\text{Corrected } \Sigma x^2}{N}} - (\text{Correct } \bar{x})^2$

$$= \sqrt{\frac{387380}{100} - (60.5)^2}$$

$$= \sqrt{3873.8 - 3660.25}$$

$$= \sqrt{213.55} = 14.6$$

(i) The set of all possible outcomes are known.

(ii) Exact outcome is not known.

Example: Tossing a coin; rolling a die

2. Sample space

The set of all possible outcomes in a random experiment is called a sample space.

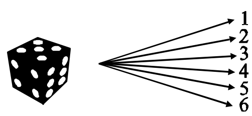
3. Sample point

Each element of a same space is called a sample point.

4. Tree diagram

Tree diagram allow us to see visually all possible outcomes of an random experiment.

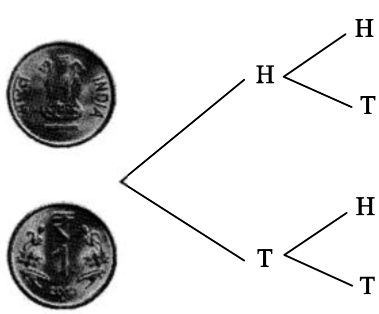
- When a dice thrown



$$S = \{ 1,2,3, 4, 5, 6 \}$$

- When we toss two coins

$$S = \{ HH, HT, TH, TT \}$$



5. Event

In a random experiment, each possible outcome is called an event.

6. Trial

Performing an experiment once is called a trial.

7. Probability of an event

$$P(E) = \frac{\text{Number of outcomes favourable to } E}{\text{Number of all possible outcomes}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

Note:

(i) $P(S) = \dfrac{n(S)}{n(S)} = 1.$

The probability of sure event is 1.

(ii) $P(Q) = \dfrac{n(Q)}{n(S)} = \dfrac{0}{n(S)} = 0.$

The probability of impossible event is 0

(iii) Probability value always lies from 0 to 1

$$0 \le P(E) \le 1$$

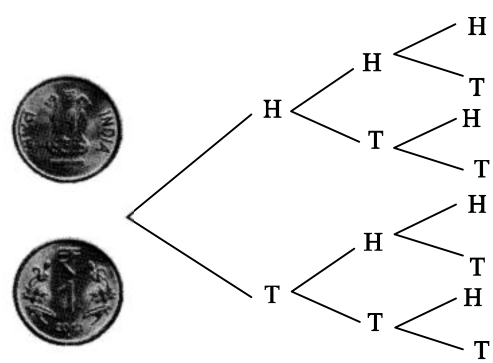
(iv) The complement event of E is $\overline{E}$

$$P(\overline{E}) = 1 - P(E)$$

(v) $P(E) + P(\overline{E}) = 1$

Type I: Sample space using [free diagram]
Q.No. 1, 2, Example 8.18

1. Write the sample space for tossing three coins using tree diagram.



Sample space

$= \{ (HHH), (HHT), (HTH), (HTT), (THH),$
$(THT), (TTH), (TTT) \}$

---

2. Write the sample space for selecting two balls from a bag containing 6 balls numbered 1 to 6 (using tree diagram).

S = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

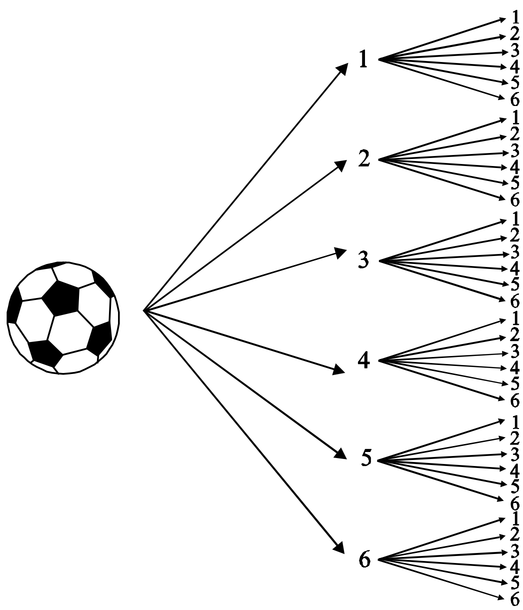
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)

(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)

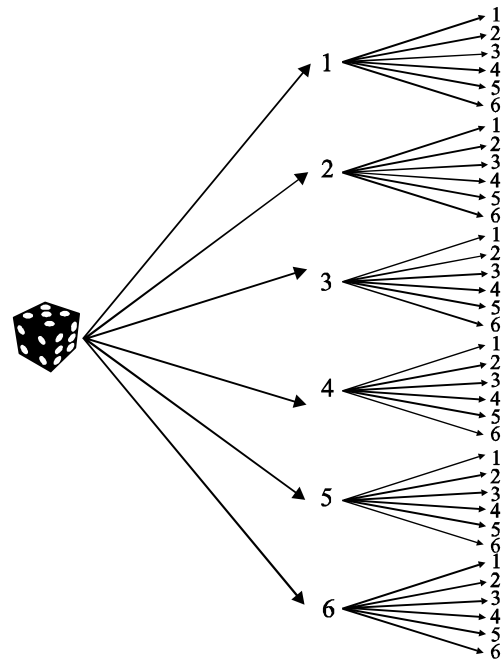
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }



Example 8.18

Express the sample space for rolling two dice using tree diagram.

When we roll two dice, since each die contain 6 faces marked with 1, 2, 3, 4, 5, 6 the tree diagram will look like



Hence, the sample space can be written as

S = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)

(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)

(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)

(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)

(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

Type II: Problems based on dice, coins and cards

Q.No. 4, 8. (i) (ii) (iii) (iv), Example 8.21, 7, Example 8.20, 9, Example 8.24, 12, Example 8.22

4. A coin is tossed thrice. What is the probability of getting two consecutive tails?

When a coin is tossed thrice,

$S = \{ (HHH), (HHT), (HTH), (HTT), (THH),$
$(THT), (TTH), (TTT) \}$

$n(S) = 8$

Let A bet the event of getting 2 tails continuously,

$A = \{ (HTT), (TTH), (TTT) \}$

$n(A) = 3$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

8. Three fair coins are tossed together. Find the probability of getting
(i) all heads
(ii) atleast one tail
(iii) almost one head
(iv) almost two tails

When 3 fair coins are tossed,

$$S = \{ (HHH), (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT) \}$$

$n(S) = 8$

(i) Let A be the event of getting all heads.

$$A = \{ (HHH) \}$$

$$n(A) = 1$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$$

(ii) Let B be the event of getting atleast one tail.

$$B = \{ (HHT), (HTH), (HTT), (THH), (THT), (TTH), (TTT) \}$$

$n(B) = 7$

$$P(B) = \frac{7}{8}$$

(iii) Let C be the event of getting at most one head.

$$C = \{ (HTT), (THT), (TTH), (TTT) \}$$

$n(C) = 4$

$$P(C) = \frac{4}{8} = \frac{1}{2}$$

(iv) Let D - almost 2 tails

$$D = \{ (HHH), (HHT), (HTT), (HTH), (THH), (THT), (TTH) \}$$

$n(D) = 7$

$$P(D) = \frac{7}{8}$$

Example 8.21

Two coins are tossed together. What is the probability of getting different faces on the coins?

When two coins are tossed together, the sample space is

$$S = \{ HH, HT, TH, TT \};\ n(S) = 4$$

Let A be the event getting different faces on the coins.

$$A = \{ HT, TH \};\ n(A) = 2$$

Probability of getting different faces on the coins is

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

7. Two unbiased dice are rolled once. Find the probability of getting
(i) a doublet (equal numbers on both dice)
(ii) the product as a prime number
(iii) the sum as a prime number
(iv) the sum as 1

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$$
$$(6,1),(6,2),(6,3), (6,4), (6,5), (6,6) \}$$

(i) Let A be the event of getting a doublet

$$A = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5),(6, 6) \}$$

$n(A) = 6$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii) Let B the event of getting the product as a prime number.

$$B = \{ (1,2), (1,3), (1,5), (2,1), (3,1), (5,1) \}$$

$n(B) = 6$

$$P(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(iii) Let C be the event of getting the sum of numbers on the dice is prime.

$$C = \{ (1,1), (1,2), (1,4), (2,1), (2,3), (2,5), (3,2), (3,4), (4,1), (4,3), (5,2), (5,6), (6, 1), (6, 5) \}$$

$$n(C) = 14$$

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{7}{36}$$

(iv) Let D be the event of getting sum of numbers is 1.

$$n(D) = 0$$

$$P(D) = 0$$

Example 8.20

Two dice are rolled. Find the probability that the sum of outcomes is (i) equal to 4 (ii) greater than 10 (iii) less than 13

When we roll two dice, the sample space is given by

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \};$$

$$n(S) = 36$$

(i) Let A be the event of getting the sum of outcome values equal to 4.

Then $A =$ (1,3), (2,2), (3,1) }; $n(A) = 3$.

Probability of getting the sum of outcomes equal to 4 is

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

(ii) Let B be the event of getting the sum of outcome values greater than 10.

Then $B = \{ (5,6), (6,5), (6,6) \}$; $n(B) = 3$

Probability of getting the sum of outcomes greater than 10 is

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

(iii) Let C be the event of getting the sum of outcomes less than 13. Here all the outcomes have the sum value less than 13.

Hence $C = S$.

Therefore, $n(C) = n(S) = 36$

Probability of getting the total value less than 13 is

$$P(C) = \frac{n(C)}{n(S)} = \frac{36}{36} = 1$$

9. Two dice are numbered 1, 2, 3, 4, 5, 6 and 1, 1, 2, 2, 3, 3 respectively. They are rolled and the sum of the numbers on them is noted. Find the probability of getting each sum from 2 to 9 separately.

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

(i) Let A - Sum of 2

$$n(A) = 2$$

$$\therefore P(A) = \frac{2}{36}$$

(ii) Let B - Sum of 3

$$n(B) = 4$$

$$P(B) = \frac{4}{36}$$

(iii) Let C - Sum of 4

$$n(C) = 6$$

$$P(C) = \frac{6}{36}$$

(iv) Let D - Sum of 5

$$n(D) = 6$$

$$P(D) = \frac{6}{36}$$

(v) Let E - Sum of 6

$$n(E) = 6$$

$$P(E) = \frac{6}{36}$$

(vi) Let F - Sum of 7

$$n(F) = 6$$

$$P(F) = \frac{6}{36}$$

(vii) Let G - Sum of 8

$$n(G) = 4$$

$$P(G) = \frac{4}{36}$$

(viii) Let H - Sum of 9

$$n(H) = 2$$

$$P(H) = \frac{2}{36}$$

Example 8.24

A die is rolled and a coin is tossed simultaneously. Find the probability that the die shows an odd number and the coin shows a head.

Sample space

$$S = \{\, 1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T \,\};$$

$$n(S) = 12$$

Let A be the event of getting an odd number and a head.

$$A = \{\, 1H, 3H, 5H \,\};\ n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{12} = \frac{1}{4}$$

12. The king and queen of diamonds, queen and jack of hearts, jack and king of spades are removed from a deck of 52 playing cards and then well shuffled. Now one card is drawn at random from the remaining cards. Determine the probability that the card is (i) a clavor (ii) a queen of red card (iii) a king of black card

By the data given

$$n(S) = 52 - 2 - 2 - 2 = 46$$

(i) Let A be the event of selecting clavor card.

$$n(A) = 13$$

$$P(A) = \frac{13}{46}$$

(ii) Let B - queen and red card.

$$n(B) = 0$$

$$P(B) = 0$$

(queen diamond and heart are included in S)

(iii) Let C - King of black cards.

$n(C) = 1$ (excluding spade king)

$$\therefore P(C) = \frac{1}{46}$$

Example 8.22

From a well shuffled pack of 52 cards, one card is draw at random. Find the probability of getting (i) red card (ii) heart card (iii) red king (iv) face card (v) number card

$$n(S) = 52$$

(i) Let A be the event of getting a red card.

$$n(A) = 26$$

Probability of getting a red card is

$$P(A) = \frac{26}{52} = \frac{1}{2}$$

(ii) Let B be the event of getting a heart card.

$$n(B) = 13$$

Probability of getting a heart card is

$$P(B) = \frac{n(B)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

(iii) Let C be the event of getting a red king card. A red king card can be either a diamond king or a heart king.

$$n(C) = 2$$

Probability of getting a red king card is

$$P(C) = \frac{n(C)}{n(S)} = \frac{2}{52} = \frac{1}{26}$$

(iv) Let D be the event of getting a face card. The face cards are Jack (J), Queen (Q), and King (K).

$$n(D) = 4 \times 3 = 12$$

Probability of getting a face card is

$$P(D) = \frac{n(D)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

(v) Let E be the event of getting a number card. The number cards are 2, 3, 4, 5, 6, 7, 8, 9 and 10.

$$n(E) = 4 \times 9 = 36$$

Probability of getting a number card is

$$P(E) = \frac{n(E)}{n(S)} = \frac{36}{52} = \frac{9}{13}$$

Type III: General probability sums

Q.No. 10, Example 8.19, 8.26, 5, 11, 15, 14, Example 8.23, 13

10. A bag contains 5 red balls, 6 white balls, 7 green balls, 8 black balls. One ball is drawn at random from the bag. Find the probability that the ball drawn is (i) white (ii) black or red (iii) not white (iv) neither white nor black.

$S = \{ 5R, 6W, 7G, 8B \}$

(i) Let A - White ball

$$n(A) = 6$$

$$P(A) = \frac{6}{26} = \frac{3}{13}$$

(ii) Let B - Black (or) red

$$n(B) = 5 + 8 = 13$$

$$P(B) = \frac{13}{26} = \frac{1}{2}$$

(iii) Let C - not white

$$n(C) = 20$$

$$P(C) = \frac{20}{26} = \frac{10}{13}$$

(iv) Let D - Neither white nor black

$$n(D) = 12$$

$$P(D) = \frac{12}{26} = \frac{6}{13}$$

Example 8.19

A bag contains 5 blue balls and 4 green balls. A ball is drawn at random from the bag. Find the probability that the ball drawn is (i) blue (ii) not blue.

Total number of possible outcomes

$$n(S) = 5 + 4 = 9$$

(i) Let A be the event of getting a blue ball.

Number of favorable outcomes for the event A. Therefore, $n(A) = 5$

Probability that the ball drawn is blue.

Therefore, $P(A) = \frac{n(A)}{n(S)} = \frac{5}{9}$

(ii) $\bar{A}$ will be the event of not getting a blue ball.

So $P(\bar{A}) = 1 - P(A) = 1 - \frac{5}{9} = \frac{4}{9}$

Example 8.26

A game of chance consists of spinning an arrow which is equally likely to come to rest pointing to one of the numbers 1, 2, 3, ... 12.

What is the probability that it will point to (i) 7 (ii) a prime number (iii) a composite number?

Sample space

$S = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}; n(S) = 12$

(i) Let A be the event of resting in 7, $n(A) = 1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{12}$$

(ii) Let B be the event that the arrow will come to rest in a prime number

$$B = \{ 2, 3, 5, 7, 11 \}; n(B) = 5$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{12}$$

(iii) Let C be the event that arrow will come to rest in a composite number.

$$C = \{ 4, 6, 8, 9, 10, 12 \}; n(C) = 6$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{12} = \frac{1}{2}$$

---

5. At a fete, cards bearing numbers 1 to 1000, one number on one card are put in a box. Each player selects one card at random and that card is not replaced. If the selected card has a perfect square number greater than 500, the player wins a prize. What is the probability that (i) the first player wins a prize (ii) the second player wins a prize, if the first has won?

$n(S) = 1000$

(i) Let A be the event of getting perfect squares between 500 and 1000

$$A = \{ 23^2, 24^2, 25^2, 26^2 \ldots 31^2 \}$$

$n(A) = 9$

$$P(A) = \frac{9}{1000}$$

is the probability for the 1st player to win a prize.

(ii) When the card which was taken first is not replaced.

$$n(S) = 999$$

$$n(B) = 8$$

$$P(B) = \frac{8}{999}$$

---

11. In a box there are 20 non-defective and some defective bulbs. If the probability that a bulb selected at random from the box found to be defective is $\frac{3}{8}$ then, find the number of defective bulbs.

Let x be the number of defective bulbs.

$$\therefore n(S) = x + 20$$

Let A be the event of selecting defective balls

$$\therefore n(A) = x$$

$$P(A) = \frac{x}{x+20}$$

Given $\frac{x}{x+20} = \frac{3}{8}$

$\Rightarrow 8x = 3x + 60$

$\Rightarrow 5x = 60$

$x = 12$

$\therefore$ The Number of defective balls = 12.

---

15. In a game, the entry fee is Rs. 150, The game consists of tossing a coin 3 times. Dhana bought a ticket for entry. If one or two heads show, she gets her entry fee back. If she throws 3 heads, she receives double the entry fees. Otherwise she will lose. Find the probability that she (i) gets double entry fee (ii) just gets her entry fee (iii) loses the entry fee.

$S = \{ (HHH), (HHT), (HTH), (THH), (HTT),$
$(THT), (HTT), (TTT) \}$

$n(S) = 8$

(i) P (gets double entry fee) $= \frac{1}{8}$ ($\because$ 3 heads)

(ii) P (just gets for her entry fee) $= \frac{6}{8} = \frac{3}{4}$

($\because$ 1 (or) 2 heads)

(iii) P (loses the entry fee) $= \frac{1}{8}$

($\because$ 3 no heads (TTT) only)

14. Two customers Priya and Amuthan are visiting a particular shop in the same week (Monday to Saturday). Each is equally likely to visit the shop on any one day as on another day. What is the probability that both will visit the shop on
(i) the same day (ii) different days
(iii) consecutive days?

Given $n(S) = 6$. (Monday - Saturday)

(i) Prob. that both of them will visit the shop on the same day $= \frac{1}{6}$

(ii) Prob. that both of them will visit the shop in different days $= \frac{5}{6}$.

($\because$ if one visits on Monday, other one visit the shop out of remaining 5 days).

(iii) Prob. that both of them will visit the shop in consecutive days.

$A = \{$ (Mon, Tue), (Tue, Wed), (Wed, Thu), (Thu, Fri), (Fri, Sat) $\}$

$n(A) = 5$

$P(A) = \frac{5}{6}$

Example 8.23

What is the probability that a leap year selected at random will contain 53 saturdays.

(Hint: $366 = 52 \times 7 + 2$)

A leap year has 366 days. So it has 52 full weeks and 2 days. 52 saturdays must be in 52 full weeks.

The possible chances for the remaining two days will be the sample space.

$S = \{$ (Sun – Mon, Mon – Tue, Tue – Wed, Wed – Thu, Thu – Fri, Fri – Sat, Sat – Sun) $\}$

$n(S) = 7$

Let A be the event of getting 53rd Saturday.

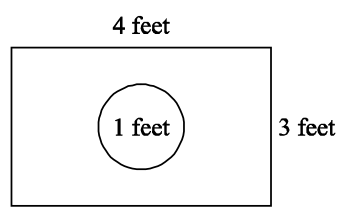
Then $A = \{$ Fri – Sat, Sat – Sun; $n(A) = 2 \}$

Probability of getting 53 Saturdays in a leap year is

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{7}$$

---

13. Some boys are playing a game, in which the stone thrown by them landing in a circular region (given in the figure) is considered as win and landing other than the circular region is considered as loss. What is the probability to win the game?



Area of the rectangular region $= 4 \times 3$
$= 12 \text{ ft}^2$

Area of the circular region $= \pi r^2$
$= \pi \times 1^2$
$= \pi \, ft^2$

$\therefore$ Probability to win the game $= \frac{\pi}{12}$
$= \frac{3.14}{12}$
$= \frac{314}{1200}$
$= \frac{157}{600}$

Type IV: Find unknown values
Q.No. 3, 6, Example 8.25

3. If A is an event of a random experiment such that $P(A): P(\overline{A}) = 17:15$ and $n(S) = 640$ then find (i) $P(\overline{A})$ (ii) $n(A)$.

Given $P(A): P(\overline{A}) = 17: 15$

$$\Rightarrow \frac{1 - P(\overline{A})}{P(A)} = \frac{17}{15}$$

$$\Rightarrow 15 - 15P(A) = 17P(\overline{A})$$

$$\Rightarrow 32P(\overline{A}) = 15$$

$$\Rightarrow P(A) = \frac{17}{32}$$

$$\Rightarrow \frac{n(A)}{n(S)} = \frac{17}{32}$$

$$\Rightarrow n(A) = \frac{17}{32} \times 640 = 340$$

---

6. A bag contains 12 blue balls and x red balls. If one ball is drawn at random (i) what is the probability tat it will be a red ball? (ii) If 8 move red balls are put in the bag, and if the probability of drawing a red ball will be twice that of the probability in (i), then find x.

Total number of balls in the bag

$= x + 12$. ($x \to$ red $12 \to$ black)

(i) Let A be the event of getting red balls

$$P(A) = \frac{n(A)}{n(S)} = \frac{x}{x+12}$$

(ii) If 8 more red balls are added in the bag.

$n(S) = x + 20$

By the problem, $\frac{x+8}{x+20} = 2\left(\frac{x}{x+12}\right)$

$$\Rightarrow (x+8)(x+12) = 2x^2 + 40x$$

$$\Rightarrow x^2 + 20x + 96 = 2x^2 + 40x$$

$$\Rightarrow x^2 + 20x - 96 = 0$$

$$\Rightarrow (x+24)(x-4) = 0$$

$$\therefore x = -24, 4$$

$$\therefore x = 4$$

$$\therefore P(A) = \frac{4}{16} = \frac{1}{4}$$

Example 8.25

A bag contains 6 green balls, some black and red balls. Number of block balls is as twice as the number of red balls. Probability of getting a green ball is thrice the probability of getting a red ball. Find (i) number of black balls and (ii) total number of balls.

Number of green balls is $n(G) = 6$

Let number of red balls is $n(R) = x$

Therefore, number of black balls is $n(B) = 2x$

Total number of balls

$$n(S) = 6 + x + 2x = 6 + 3x$$

It is given that, $P(G) = 3 \times P(R)$

$$\frac{6}{6+3x} = 3 \times \frac{x}{6+3x}$$

$3x = 6$ gives, $x = 2$.

(i) Number of black balls $= 2 \times 2 = 4$

(ii) Total number of balls $= 6 + (3 \times 2) = 12$

Exercise 8.4

KEYPOINTS

1. If A and B are two events associated with a random experiment, then

(i) $P(A \cap \overline{B}) = P(\text{only } A) = P(A) - P(A \cap B)$

(ii) $P(\overline{A} \cap B) = P(\text{only } B) = P(B) - P(A \cap B)$

2. Addition theorem of probability

(i) If A and B are any two events then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(ii) If A, B and C are any three events then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Note

(i) If A, B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

(ii) $A \cap \overline{A} = \varphi$ and $A \cup \overline{A} = S$

(iii) P (Union of mutually exclusive event)

$= \Sigma$ (Probability of events)

Type I: Simple problems based on probability formulae

Q.No.1, Example 8.27, 2, 3, Example 8.30, 4, 5

1. If $P(A) = \frac{2}{3}, P(B) = \frac{2}{5}, P(A \cup B) = \frac{1}{3}$ then find $P(A \cap B)$.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{2}{3} + \frac{2}{5} - \frac{1}{3}$$

$$= \frac{10 + 6 - 5}{15}$$

$$= \frac{11}{15}$$

Example 8.27

If $P(A) = 0.37, P(B) = 0.42, P(A \cap B) = 0.09$ then find $P(A \cup B)$.

$P(A) = 0.37, P(B) = 0.42, P(A \cap B) = 0.09$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.37 + 0.42 - 0.09 = 0.7$

2. A and B are two events such that, $P(A) = 0.42, P(B) = 0.48,$ and $P(A \cap B) = 0.16$.
Find (i) P, (not A) (ii) P, (not B) (iii) P, (A or B)

(a) P (not A) $= P(\overline{A}) = 1 - P(A)$

$= 1 - 0.42$

$= 0.58$

(b) P (not B) $= P(\overline{B}) = 1 - P(B)$

$= 1 - 0.48$

$= 0.52$

(c) P (A or B) $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= 0.42 + 0.48 - 0.16$

$= 0.74$

3. If A and B are two mutually exclusive events of a random experiment and P (not A) $= 0.45, P(A \cup B) = 0.65,$ then find $P(B)$.

Given A and B are mutually exclusive events

$$P(A \cap B) = 0$$

Also, P (not A) $= 0.45$

$\therefore P(\overline{A}) = 0.45$

$1 - P(A) = 0.45$

$P(A) = 0.55$

$P(A \cup B) = P(A) + P(B)$

$\therefore P(B) = P(A \cup B) - P(A)$

$= 0.65 - 0.55$

$= 0.10$

Example 8.30

If A and B are two events such that $P(A) = \frac{1}{4}, P(B) = \frac{1}{2}$ and P (A and B) $= \frac{1}{8}$, find (i) P (A or B) (ii) P (not A and not B).

(i) P (A or B) $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

P (A or B) $= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}$

(ii) P (not A and not B) $= P(\overline{A} \cap \overline{B})$

$= P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

P (not A and not B) $= 1 - \frac{5}{8} = \frac{3}{8}$

4. The probability that atleast one of A and B occur is 0.6. If A and B occur simultaneously with probability 0.2, then find $P(\overline{A}) + P(\overline{B})$.

Given $P(A \cup B) = 0.6, P(A \cap B) = 0.2$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.6 = P(A) + P(B) - 0.2$

$\therefore P(A) + P(B) = 0.8$

$\therefore P(\overline{A}) + P(\overline{B})$

$= 1 - P(A) + 1 - P(B)$

$= 2 - (P(A) + P(B))$

$= 2 - 0.8$

$= 1.2$

---

5. The probability of happening of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then find the probability that neither A nor B happen.

Given $P(A) = 0.5, P(B) = 0.3, P(A \cap B) = 0$

P (neither A nor B)

$= P(\overline{A} \cap \overline{B})$

$= P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - (0.8)$

$= 0.2$

Type II: Problems based on Addition theorem of probability

Q.No.6, Example 8.29, 9, 12, 7, Example 8.28, 8.31, 8, 10, 11, 14, 13

6. Two dice are rolled one. Find the probability of getting an even number on the first die or a total of face sum 8.

$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$n(S) = 36$

Let A be the event of getting even number on the 1^{st} die.

$A = \{ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$n(A) = 18$

$P(A) = \frac{18}{36}$

Let B - Total of face sum as 8.

$B = \{ (2,6),(3,5),(4,4),(5,3),(6,2) \}$

$n(B) = 5, P(B) = \frac{5}{36}$

$A \cap B = \{ (2,6), (4,4), (6,2) \}$

$n(A \cap B) = 3$

$P(A \cap B) = \frac{3}{36}$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{18}{36} + \frac{5}{36} - \frac{3}{36}$

$= \frac{20}{36}$

$= \frac{5}{9}$

Example 8.29

Two dice are rolled together. Find the probability of getting a doublet or sum of faces as 4.

When two dice are rolled together, there will be $6 \times 6 = 36$ outcomes. Let S be the sample space. Then $n(S) = 36$

Let A be the event of getting a doublet and B be the event of getting face sum 4.

Then $A = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6)$

$B = \{ (1,3), (2,2), (3,1) \}$

Therefore, $A \cap B = \{ (2,2) \}$

Then, $n(A) = 6, n(B) = 3, n(A \cap B) = 1.$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{36}$$

Therefore,

P (getting a doublet or a total of 4) $= P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{36} + \frac{3}{36} - \frac{1}{36} = \frac{8}{36} = \frac{2}{9}$$

Hence, the required probability is $\frac{2}{9}$

---

9. Three unbiased coins are tossed once. Find the probability of getting atmost 2 tails or atleast 2 heads.

$S = \{ (HHH), (HHT), (HTH), (THH), (HTT),$
$(THT), (TTH), (TTT) \}$

$n(S) = 8$

Let A - at most 2 tails

$A = \{ (HHT), (HTH), (THH), (HTT), (THT),$
$(TTH), (HHH) \}$

$n(A) = 7$

$P(A) = \frac{7}{8}$

Let B - atleast 2 heads

$B = \{ (HHH), (HHT), (HTH), (THH) \}$

$n(B) = 4$

$P(B) = \frac{4}{8}$

$\therefore A \cap B = \{ (HHH), (HHT), (HTH), (THH) \}$

$n(A \cap B) = 4, P(A \cap B) = \frac{4}{8}$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{7}{8} + \frac{4}{8} - \frac{4}{8}$$

$$= \frac{7}{8}$$

---

12. A coin is tossed thrice. Find the probability of getting exactly two heads or atleast one tail or two consecutive heads.

$S = \{ (HHH), (HHT), (HTH), (THH), (TTH),$
$(THT), (HTT), (TTT) \}$

$n(S) = 8$

Let A - exactly 2 heads

$A = \{ (HHT), (HTH), (THH) \}$

$n(A) = 3$

$P(A) = \frac{3}{8}$

Let B - atleast one tail

$B = \{ (HHT), (HTH), (THH), (TTH), (THT),$
$(HTT), (TTT) \}$

$n(B) = 7$

$P(B) = \frac{7}{8}$

Let C - Consecutively 2 heads

$C = \{ (HHH), (HHT), (THH) \}$

$n(C) = 3$

$P(C) = \frac{3}{8}$

$A \cap B = \{ (HHT), (HTH), (THH) \}$

$n(A \cap B) = 3$

$P(A \cap B) = \frac{3}{8}$

$B \cap C = \{ (HHT), (THH) \}$

$n(B \cap C) = 2$

$$P(B \cap C) = \frac{2}{8}$$

$$C \cap A = \{ (HHT), (THH) \}$$

$$n(C \cap A) = 2$$

$$P(C \cap A) = \frac{2}{8}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{3}{8} + \frac{7}{8} + \frac{3}{8} - \frac{3}{8} - \frac{2}{8} - \frac{2}{8} + \frac{2}{8}$$

$$= \frac{8}{8} = 1$$

7. From a well-shuffled pack of 52 cards, a card is drawn at random. Find the probability of it being either a red king or a black queen.

$n(S) = 52$

Let A - Red King

$n(A) = 2$

$$P(A) = \frac{2}{52}$$

Let B - Black Queen

$$n(B) = 2$$

$$P(B) = \frac{2}{52}$$

Here A and B mutually exclusive

$$\therefore P(A \cup B) = P(A) + P(B)$$

$$= \frac{4}{52}$$

$$= \frac{1}{13}$$

Example 8.28

What is the probability of drawing either a king or a queen in a single draw from a well shuffled pack of 52 cards?

Total number of cards = 52

Number of king cards = 4

Probability of drawing a king card $= \frac{4}{52}$

Number of queen cards = 4

Probability of drawing a queen card $= \frac{4}{52}$

Both the events of drawing a king and a queen are mutually exclusive

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$

Therefore, probability of drawing either a king or a queen $= \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$

Example 8.31

A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Total number of cards = 52; $n(S) = 52$

Let A be the event of getting a king card.

$n(A) = 4$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

Let B be the event of getting a heart card.

$n(B) = 13$

$$P(B) = \frac{n(B)}{n(S)} = \frac{13}{52}$$

Let C be the event of getting a red card.

$n(C) = 26$

$$P(C) = \frac{n(C)}{n(S)} = \frac{26}{52}$$

$$P(A \cap B) = P(\text{getting heart king}) = \frac{1}{52}$$

$$P(B \cap C) = P(\text{getting red and heart}) = \frac{13}{52}$$

$$P(B \cap C) = P(\text{getting red king}) = \frac{2}{52}$$

$$P(A \cap B \cap C) = P(\text{getting heart, king which is red})$$

$$= \frac{1}{52}$$

Therefore, required probability is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}$$

---

8. A box contains cards numbered 3, 5, 7, 9, … 35, 37. A card is drawn at random from the box. Find the probability that the drawn card have either multiples of 7 or a prime number.

$$S = \{ 3, 5, 7, 9, \ldots 35, 37 \}$$

$$n(S) = 18$$

Let A - multiple of 7.

$$A = \{ 7, 14, 21, 28, 35 \}$$

$$n(A) = 5$$

$$P(A) = \frac{5}{18}$$

Let B - a prime number

$$B = \{ 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37 \}$$

$$n(B) = 11$$

$$P(B) = \frac{11}{18}$$

Here $A \cap B = \{ 7 \}$

$$n(A \cap B) = 1$$

$$P(A \cap B) = \frac{1}{18}$$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{5}{18} + \frac{11}{18} - \frac{1}{18}$$

$$= \frac{15}{18}$$

$$= \frac{5}{6}$$

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10. The probability that a person will get an electrification contract is $\frac{3}{5}$ and the probability that he will not get plumbing contract is $\frac{5}{8}$. The probability of getting atleast one contract is $\frac{5}{7}$. What is the probability that he will get both?

Let A - electrification contract

B - not plumbing contract

Given

$$P(A) = \frac{3}{5}, P(\overline{B}) = \frac{5}{8}, P(A \cup B) = \frac{5}{7}$$

$$\Rightarrow P(B) = 1 - \frac{5}{8}$$

$$= \frac{3}{8}$$

$$\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{3}{5} + \frac{3}{8} - \frac{5}{7}$$

$$= \frac{168 + 15 - 200}{280}$$

$$= \frac{73}{280}$$

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11. In a town of 8000 people, 1300 are over 50 years and 3000 are females. It is know that 30% of the females are over 50 years. What is the probability that a chosen individual from the town is either a female or over 50 years?

Let A - Female

B - Over 50 years

Given $n(S) = 8000, n(A) = 3000,$

$n(B) = 1300$ and

$$n(A \cap B) = \frac{30}{100} \times 3000 = 900$$

$$\therefore P(A) = \frac{3000}{8000}, P(B) = \frac{1300}{8000}, P(A \cap B) = \frac{900}{8000}$$

$\therefore P$ (either a female (or) over 50 years)

$$P(A \cap B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3000 + 1300 - 900}{8000}$$

$$= \frac{3400}{8000}$$

$$= \frac{34}{80}$$

$$= \frac{17}{40}$$

14. In a class of 35, students are numbered from 1 to 35. The ratio of boys to girls is 4:3. The roll numbers of students begin with boys and end with girls. Find the probability that a student selected is either a boy with prime roll number or a girl with composite roll number or an even roll number.

Given $n(S) = 35$ and ratio of boys and girls = 4:3

No. of boys $= \frac{4}{7} \times 35 = 20$

No. of boys $= \frac{3}{7} \times 35 = 15$

Let A - a boy with prime roll no

$A = \{ 2, 3, 5, 7, 11, 13, 19 \}$

($\because$ only 20 boys)

$n(A) = 7$

$P(A) = \frac{7}{35}$

Let B - a girl with composite roll no.

$B = \{ 21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35 \}$

$n(B) = 12$

$\therefore P(B) = \frac{12}{25}$

Let C - even roll no.

$B = \{ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34 \}$

$n(C) = 17$

$\therefore P(C) = \frac{17}{35}$

$A \cap B = \{ \ \}, n(A \cap B) = 0, P(A \cap B) = 0$

$B \cap C = \{ 22, 24, 26, 28, 30, 32, 34 \}$

$\therefore n(B \cap C) = 7 \Rightarrow P(B \cap C) = \frac{7}{35}$

$C \cap A = \{ 2 \} \cap n(C \cap A) = 1$

$P(C \cap A) = \frac{1}{35}$

at $P(A \cap B \cap C) = 0$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{7}{35} + \frac{12}{35} + \frac{17}{35} - 0 - \frac{7}{35} - \frac{1}{35} + 0$$

$$= \frac{28}{35}$$

$$= \frac{4}{5}$$

Type III: Only *A* and only *B* based sums

Example 8.32, 8.33

Example 8.32

In a class of 50 students, 28 opted for NCC, 30 opted for NSS and 18 opted both NCC and NSS. One of the students is selected at random. Find the probability that

(i) The student opted for NCC but not NSS.

(ii) The student opted for NSS but not NCC.

(iii) The student opted for exactly one of them.

Total number of students $n(S) = 50$.

Let A and B be the events of students opted for NCC and NSS respectively.

$n(A) = 28, n(B) = 30, n(A \cap B) = 18$

$$P(A) = \frac{n(A)}{n(S)} = \frac{28}{50}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{30}{50}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{18}{50}$$

(i) Probability of the students opted for NCC but not NSS

$$P(A \cap \overline{B}) = P(A) - P(A \cap B) = \frac{28}{50} - \frac{18}{50} = \frac{1}{5}$$

(ii) Probability of the students opted for NSS but not NCC

$$P(A \cap \overline{B}) = P(B) - P(A \cap B) = \frac{30}{50} - \frac{18}{50} = \frac{6}{25}$$

(iii) Probability of the students opted for exactly one of them

$$= P[(A \cap \overline{B}) \cup (\overline{A} \cap B)]$$

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B) = \frac{1}{5} + \frac{6}{25} = \frac{11}{25}$$

(Note that $(A \cap \overline{B}), (\overline{A} \cap B)$ are mutually exclusive events)

Example 8.33

A and B are two candidates seeking admission to IIT. The probability that A getting selected is 0.5 and the probability that both A and B getting selected is 0.3. Prove that the probability of B being selected is atmost 0.8.

$P(A) = 0.5, P(A \cap B) = 0.3$

We have $P(A \cup B) \leq 1$

$P(A) + P(B) - P(A \cap B) \leq 1$

$0.5 + P(B) - 0.3 \leq 1$

$P(B) \leq 1 - 0.2$

$P(B) \leq 0.8$

Therefore, probability of B getting selected is atmost 0.8.

Exercise 8.5

Multiple choice Questions

1. Which of the following is not a measure of dispersion?

1. Range
2. Standard deviation
3. Arithmetic mean
4. Variance

Ans. (3) Arithmetic mean

2. The range of the data 8, 8, 8, 8, 8 … 8 is

1. 0 2. 1
3. 8 4. 3

Range of equal values is 0 **Ans.** (1) 0

3. The sum of all deviations of the data from its mean is

1. Always positive
2. Always negative
3. Zero
4. Non-zero integer

We know that $\Sigma(x - \overline{x}) = 0$ **Ans.** (3) Zero

4. The mean of 100 observations is 40 and their standard deviation is 3. The sum of squares of all deviation is

1. 40000 2. 160900
3. 160000 4. 30000

$n = 100$

$\overline{x} = 40$

$\sigma = 3$

We know that

$$\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$$

$$3^2 = \frac{\Sigma x^2}{100} - (40)^2$$

$$9 = \frac{\Sigma x^2}{100} - 1600$$

$$9 + 1600 = \frac{\Sigma x^2}{100}$$